目　录

一、常数$C(a,k)$的定义及其性质

定理1 设a与k都是正整数，则

$$\frac{1}{k}\left[\ln(nk+a)-\ln a\right]<\sum_{i=1}^{n}\frac{1}{k(i-1)+a}$$

$$\leqslant\frac{1}{a}+\frac{1}{k}\left[\ln(nk-k+a)-\ln a\right].$$

证明 数列$\left\{\left(1+\frac{1}{n}\right)^{n}\right\}$单调递增而趋于$e$，所以

$$\left(1+\frac{1}{n}\right)^{n}<e,\quad n\ln\frac{n+1}{n}<1,$$

$$\ln(n+1)-\ln n<\frac{1}{n}.\ (n=1,\ 2,\ \cdots)$$

令 $n=i,\ i+1,\ \cdots i+k-1$，然后各式相加得：

$$\ln(i+k)-\ln i<\frac{1}{i}+\frac{1}{i+1}+\cdots+\frac{1}{i+k-1}\leqslant\frac{k}{i},$$

$$\frac{1}{k}\left[\ln(i+k)-\ln i\right]<\frac{1}{i}.\ (i=1,2,\cdots)$$

令$i=a,\ k+a,\ 2k+a,\ \cdots k(n-1)+a$，然后各式相加得：

$$\frac{1}{k}\left[\ln(nk+a)-\ln a\right]<\sum_{i=1}^{n}\frac{1}{k(i-1)+a}.$$

数列$\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$单调递减而趋于$e$，所以

$$e<\left(1+\frac{1}{n}\right)^{n+1},\quad 1<(n+1)\ln\frac{n+1}{n},$$

$$\frac{1}{n+1}<\ln(n+1)-\ln n.\quad(n=1,\ 2,\ \cdots)$$

令　$n=i,\ i+1,\ \cdots i+k-1$，然后各式相加得

$$\frac{1}{i+1}+\frac{1}{i+2}+\cdots+\frac{1}{i+k}<\ln(i+k)-\ln i,$$

$$\frac{1}{i+k}<\frac{1}{k}\Big[\ln(i+k)-\ln i\Big].\quad(i=1,\ 2,\ \cdots)\qquad(*)$$

令 $i=a,\ k+a,\ 2k+a,\ \cdots k(n-2)+a$，然后各式相加得

$$\sum_{i=1}^{n}\frac{1}{k(i-1)+a}<\frac{1}{k}\Big[\ln(nk-k+a)-\ln a\Big].$$

于是 $\displaystyle\sum_{i=1}^{n}\frac{1}{k(i-1)+a}\leqslant\frac{1}{a}+\frac{1}{k}\Big[\ln(nk-k+a)-\ln a\Big].$

定理2　设 a 与 k 都是正整数，则

$$\sum_{i=1}^{n}\frac{1}{k(i-1)+a}=C(a,k)+\frac{1}{k}\ln\frac{k(n-1)+a}{a}+\varepsilon_n(a,k),$$

其中 $C(a,k)$ 是常数，当 $n\to\infty$ 时 $\varepsilon_n(a,k)\to 0$。

为了方便，记 $H_n(a,k)=\displaystyle\sum_{i=1}^{n}\frac{1}{k(i-1)+a}$，则

$$H_n(a,k)=C(a,k)+\frac{1}{k}\ln\frac{k(n-1)+a}{a}+\varepsilon_n(a,k).$$

证明　于$(*)$令 $i=nk-k+a$，则有

$$\frac{1}{nk+a}<\frac{1}{k}\ln\frac{nk+a}{nk-k+a}.\quad(n=1,2,\cdots)$$

令　$X_n=H_n(a,k)-\dfrac{1}{k}\ln\dfrac{k(n-1)+a}{a}$，

则 $X_{n+1}=H_{n+1}(a,k)-\frac{1}{k}\ln\frac{nk+a}{a}$。

于是 $X_{n+1}-X_n=\frac{1}{nk+a}-\frac{1}{k}\ln\frac{nk+a}{nk-k+a}<0$。

$(n=1,2,\cdots)$

根据定理1，有

$$X_n=H_n(a,k)-\frac{1}{k}\ln\frac{k(n-1)+a}{a}$$

$$>\frac{1}{k}\ln\frac{nk+a}{nk-k+a}>0。(n=1,2,\cdots)$$

已证明数列 $\{X_n\}$ 单调递减而且下方有界，所以极限 $\lim\limits_{n\to\infty}X_n$ 存在。设 $\lim\limits_{n\to\infty}X_n=C(a,k)$，即

$$\lim_{n\to\infty}\left[H_n(a,k)-\frac{1}{k}\ln\frac{k(n-1)+a}{a}\right]$$

$$=C(a,k), \qquad (**)$$

则 $H_n(a,k)=C(a,k)+\frac{1}{k}\ln\frac{k(n-1)+a}{a}+\varepsilon_n(a,k)$，

当 $n\to\infty$ 时 $\varepsilon_n(a,k)\to 0$。

上述极限$(**)$就是常数 $C(a,k)$ 的定义。

常数 $C(a,k)$ 具有如下的两个性质：

命题1 设 m 为正整数，则

$$C(ma,mk)=\frac{1}{m}C(a,k)。$$

证明 $H_n(ma,mk)=\frac{1}{m}H_n(a,k)$。

根据定理2，有

$$C(ma,mk)+\frac{1}{mk}\ln\frac{mk(n-1)+ma}{ma}+\varepsilon_n(ma,mk)$$

$$=\frac{1}{m}\left[C(a,k)+\frac{1}{k}\ln\frac{k(n-1)+a}{a}+\varepsilon_n(a,k)\right].$$

于是 $C(ma,mk)+\varepsilon_n(ma,mk)$

$$=\frac{1}{m}C(a,k)+\frac{1}{m}\varepsilon_n(a,k).$$

令 $n\to\infty$ 并取极限，命题即得证。

命题2 设 b 为正整数，则

$$C(a+bk,k)=C(a,k)+\frac{1}{k}\ln\frac{a+bk}{a}$$
$$-\left[\frac{1}{a}+\frac{1}{k+a}+\frac{1}{2k+a}+\cdots+\frac{1}{(b-1)k+a}\right].$$

证明 根据定理2，有

$$\frac{1}{a}+\frac{1}{k+a}+\frac{1}{2k+a}+\cdots+\frac{1}{(b-1)k+a}$$
$$-\left[\frac{1}{kn+a}+\frac{1}{k(n+1)+a}+\cdots+\frac{1}{k(n-1+b)+a}\right]$$
$$=H_n(a,k)-H_n(a+bk,k)$$
$$=C(a,k)-C(a+bk,k)+\frac{1}{k}\ln\frac{nk-k+a}{nk-k+bk+a}$$
$$+\frac{1}{k}\ln\frac{a+bk}{a}+\varepsilon_n(a,k)-\varepsilon_n(a+bk,k).$$

令 $m\to\infty$ 并取极限，命题即得证。

定理3 设 A 为非负整数，a，b，B，k 都是正整数，且 $b\geqslant 2$，则

$$\lim_{n\to\infty}\left[\frac{1}{k(Bn+A)+a}+\frac{1}{k(Bn+B+A)+a}\right.$$
$$\left.+\frac{1}{k(Bn+2B+A)+a}+\cdots+\frac{1}{k(bBn-B+A)+a}\right]$$
$$=\frac{1}{Bk}\ln b.$$

证明　根据定理2，有

$$\frac{1}{k(Bn+A)+a}+\frac{1}{k(Bn+B+A)+a}$$

$$+\frac{1}{k(Bn+2B+A)+a}+\cdots+\frac{1}{k(bBn-B+A)+a}$$

$$=H_{bn}(a+Ak,Bk)-H_n(a+Ak,Bk)$$

$$=\frac{1}{Bk}\ln\frac{k(bBn-B+A)+a}{k(An-B+B)+a}+\varepsilon_{bn}(a+Ak,Bk)$$

$$-\varepsilon_n(a+Ak,Bk).$$

令 $n\to\infty$ 并取极限，定理即得证。

令 $A=0$，$B=1$，则有

$$\lim_{n\to\infty}\left[\frac{1}{kn+a}+\frac{1}{k(n+1)+a}+\frac{1}{k(n+2)+a}\right.$$

$$\left.+\cdots+\frac{1}{k(bn-1)+a}\right]=\frac{1}{k}\ln b.$$

于此令 $a=1$，$b=2$，$k=1$，即得通常所见的极限

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\right)=\ln 2.$$

定理4　设 m 为正整数，且 $m\geqslant 2$，则

$$C(a,mk)+C(a+k,mk)+C(a+2k,mk)$$

$$+\cdots+C(a+mk-k,mk)-C(a,k)$$

$$=\frac{1}{mk}\ln\frac{(a+k)(a+2k)\cdots(a+mk-k)}{a^{m-1}}.$$

证明　根据定理2，有

$$\frac{1}{kn+a}+\frac{1}{k(n+1)+a}+\frac{1}{k(n+2)+a}+\cdots$$

$$+\frac{1}{k(mn-1)+a}$$

$$
\begin{aligned}
&= H_n(a,mk) + H_n(a+k,mk) + H_n(a+2k,mk) \\
&\quad + \cdots + H_n(a+mk-k,mk) - H_n(a,k) \\
&= C(a,mk) + C(a+k,mk) + C(a+2k,mk) \\
&\quad + \cdots + C(a+mk-k,mk) - C(a,k) \\
&\quad + \frac{1}{mk} \ln \frac{(mnk-mk+a)\cdots(mnk-k+a)}{(nk-k+a)^m} \\
&\quad - \frac{1}{mk} \ln \frac{a(a+k)(a+2k)\cdots(a+mk-k)}{a^m} \\
&\quad + \varepsilon_n(a,mk) + \varepsilon_n(a+k,mk) + \varepsilon_n(a+2k,mk) \\
&\quad + \cdots + \varepsilon_n(a+mk-k,mk) - \varepsilon_n(a,k).
\end{aligned}
$$

注意到定理3，令 $n \to \infty$ 并取极限，定理即得证。

二、常数$C(a,k)$与$A(a,k)$的关系

记 $A(a,k)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{k(n-1)+a}$.

定理5 $2C(a,2k)-C(a,k)=A(a,k)$。

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{1}{k(2i-2)+a}-\frac{1}{k(2i-1)+a}\right]$$

$$=H_n(a,2k)-H_n(a+k,2k)$$

$$=C(a,2k)-C(a+k,2k)+\frac{1}{2k}\ \ln\frac{2nk-2k+a}{2nk-k+a}$$

$$+\frac{1}{2k}\ \ln\frac{a+k}{a}+\varepsilon_n(a,2k)-\varepsilon_n(a+k,2k).$$

令 $n\to\infty$ 并取极限即得

$$A(a,k)=\sum_{n=1}^{\infty}\left[\frac{1}{k(2n-2)+a}-\frac{1}{k(2n-1)+a}\right]$$

$$=C(a,2k)-C(a+k,2k)+\frac{1}{2k}\ \ln\frac{a+k}{a}. \qquad (1)$$

于定理4令 $m=2$，则有

$$C(a,2k)+C(a+k,2k)-C(a,k)$$

$$=\frac{1}{2k}\ \ln\frac{a+k}{a}, \qquad (2)$$

$$2C(a,2k)-C(a,k)$$

$$=C(a,2k)-C(a+k,2k)+\frac{1}{2k}\ln\frac{a+k}{a}.$$

于是 $2C(a,2k)-C(a,k)=A(a,k)$。

例 求证 $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\ln 2$。

证明 由(2)得

$$C(a,2k)-C(a+k,2k)+\frac{1}{2k}\ln\frac{a+k}{a}$$

$$=C(a,k)-2C(a+k,2k)+\frac{1}{k}\ln\frac{a+k}{a}.$$

于是由(1)得

$$A(a,k)$$

$$=C(a,k)-2C(a+k,2k)+\frac{1}{k}\ln\frac{a+k}{a}.$$

根据命题1，$2C(2,2)=C(1,1)$。于上式令 $a=1$，$k=1$ 即得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=C(1,1)-2C(2,2)+\ln 2=\ln 2.$$

已求得 $A(1,1)=\ln 2$，应用积分方法容易求得

$$A(1,2)=\int_0^1\frac{dx}{x^2+1}=\frac{\pi}{4}.$$

仿此求得

$$A(1,3)=\frac{\pi}{3\sqrt{3}}+\frac{1}{3}\ln 2;$$

$$A(2,3)=\frac{\pi}{3\sqrt{3}}-\frac{1}{3}\ln 2;$$

$$A(5,3)=-A(2,3)+\frac{1}{2}=-\frac{\pi}{3\sqrt{3}}+\frac{1}{3}\ln 2+\frac{1}{2};$$

$$A(1,4)=\frac{\pi}{4\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1);$$

$$A(3,4)=\frac{\pi}{4\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1);$$

$$A(5,4)=-A(1,4)+1$$

$$=-\frac{\pi}{4\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+1;$$

$$A(7,4)=-A(3,4)+\frac{1}{3}$$

$$=-\frac{\pi}{4\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{3};$$

$$A(1,6)=\frac{\pi}{6}+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3});$$

$$A(5,6)=\frac{\pi}{6}-\frac{1}{2\sqrt{2}}\ln(2+\sqrt{3});$$

$$A(7,6)=-A(1,6)+1$$

$$=-\frac{\pi}{6}-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3})+1;$$

$$A(11,6)=-A(5,6)+\frac{1}{5}$$

$$=-\frac{\pi}{6}+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{5}.$$

定理6 $2C(1,2)=C(1,1)+\ln 2;$

$$3C(1,3)=C(1,1)+\frac{\pi}{2\sqrt{3}}+\frac{1}{2}\ln 3;$$

$$3C(2,3)=C(1,1)-\frac{\pi}{2\sqrt{3}}+\frac{1}{2}\ln 12;$$

$$4C(1,4)=C(1,1)+\frac{\pi}{2}+\ln 2;$$

$$4C(3,4)=C(1,1)-\frac{\pi}{2}+\ln 6;$$

$$6C(1,6)=C(1,1)+\frac{\pi\sqrt{3}}{2}+\frac{1}{2}\ln 12;$$

$$6C(5,6)=C(1,1)-\frac{\pi\sqrt{3}}{2}+\frac{1}{2}\ln 300;$$

$$8C(1,8) = C(1,1) + \frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 2$$
$$+\sqrt{2}\ln(\sqrt{2}+1);$$

$$8C(3,8) = C(1,1) + \frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 6$$
$$-\sqrt{2}\ln(\sqrt{2}+1);$$

$$8C(5,8) = C(1,1) - \frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 10$$
$$-\sqrt{2}\ln(\sqrt{2}+1);$$

$$8C(7,8) = C(1,1) - \frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 14$$
$$+\sqrt{2}\ln(\sqrt{2}+1);$$

$$12C(1,12)=C(1,1)+\frac{\pi\sqrt{3}}{2}+\pi+\frac{1}{2}\ln 12$$
$$+\sqrt{3}\ln(2+\sqrt{3});$$

$$12C(5,12)=C(1,1)-\frac{\pi\sqrt{3}}{2}+\pi+\frac{1}{2}\ln 300$$
$$-\sqrt{3}\ln(2+\sqrt{3});$$

$$12C(7,12)=C(1,1)+\frac{\pi\sqrt{3}}{2}-\pi+\frac{1}{2}\ln 7^2\cdot 12$$
$$-\sqrt{3}\ln(2+\sqrt{3});$$

$$12C(11,12)=C(1,1)-\frac{\pi\sqrt{3}}{2}-\pi+\frac{1}{2}\ln 11^2\cdot 12$$
$$+\sqrt{3}\ln(2+\sqrt{3}).$$

证明 于定理5令 $a=1$，$k=1$ 即得

$$2C(1,2)=C(1,1)+A(1,1)=C(1,1)+\ln 2.$$

于定理5令 $a=2$，$k=3$ 即得

$$C(1,3)-C(2,3)=A(2,3)=\frac{\pi}{3\sqrt{3}}-\frac{1}{3}\ln 2.$$

于定理4令 $a=1$，$k=1$，$m=3$ 即得

$$C(1,3)+C(2,3)+\frac{1}{3}C(1,1)-C(1,1)=\frac{1}{3}\ln 6.$$

由此即可求得

$$3C(1,3)=C(1,1)+\frac{\pi}{2\sqrt{3}}+\frac{1}{2}\ln 3,$$

$$3C(2,3)=C(1,1)-\frac{\pi}{2\sqrt{3}}+\frac{1}{2}\ln 12.$$

于定理5令 $a=1$，$k=2$ 即得

$$2C(1,4)=C(1,2)+A(1,2),$$

于是
$$\begin{aligned}4C(1,4)&=2C(1,2)+2A(1,2)\\&=C(1,1)+\frac{\pi}{2}+\ln 2.\end{aligned}$$

于(1)令 $a=1$，$k=2$ 即得

$$C(1,4)-C(3,4)+\frac{1}{4}\ln 3=A(1,2)=\frac{\pi}{4},$$

于是 $4C(3,4)=4C(1,4)-\pi+\ln 3=C(1,1)-\dfrac{\pi}{2}+\ln 6.$

于定理5令 $a=1$，$k=3$ 即得

$$2C(1,6)-C(1,3)=A(1,3)=\frac{\pi}{3\sqrt{3}}+\frac{1}{3}\ln 2,$$

于是 $6C(1,6)=3C(1,3)+\dfrac{\pi}{\sqrt{3}}+\ln 2$

$$= C(1,1) + \frac{\pi\sqrt{3}}{2} + \frac{1}{2}\ln 12.$$

于定理5令 $a=5$, $k=3$ 即得

$$2C(5,6) - C(5,3)$$
$$= A(5,3) = -\frac{\pi}{3\sqrt{3}} + \frac{1}{3}\ln 2 + \frac{1}{2}.$$

于命题2令 $a=2$, $b=1$, $k=3$ 即得

$$C(5,3) = C(2,3) + \frac{1}{3}\ln\frac{5}{2} - \frac{1}{2}.$$

于是 $$2C(5,6) = C(2,3) - \frac{\pi}{3\sqrt{3}} + \frac{1}{3}\ln 5,$$

$$6C(5,6) = 3C(2,3) - \frac{\pi}{\sqrt{3}} + \ln 5$$
$$= C(1,1) - \frac{\pi\sqrt{3}}{2} + \frac{1}{2}\ln 300.$$

于定理5令 $a=1$, $k=4$ 即得

$$2C(1,8) - C(1,4)$$
$$= A(1,4) = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

于是 $$8C(1,8) = 4C(1,4) + \frac{\pi}{\sqrt{2}} + \sqrt{2}\ln(\sqrt{2}+1)$$
$$= C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 2$$
$$+ \sqrt{2}\ln(\sqrt{2}+1).$$

于定理5令 $a=3$, $k=4$ 即得

$$2C(3,8) - C(3,4) = A(3,4)$$
$$= \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

于是 $8C(3,8)=4C(3,4)+\dfrac{\pi}{\sqrt{2}}-\sqrt{2}\ln(\sqrt{2}+1)$

$$=C(1,1)+\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 6$$

$$-\sqrt{2}\ln(\sqrt{2}+1).$$

于定理5令 $a=5$，$k=4$ 即得

$$2C(5,8)-C(5,4)$$

$$=A(5,4)=-\frac{\pi}{4\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+1.$$

于命题2令 $a=1$，$b=1$，$k=4$ 即得

$$C(5,4)=C(1,4)+\frac{1}{4}\ln 5-1.$$

于是 $2C(5,8)=C(1,4)-\dfrac{\pi}{4\sqrt{2}}+\dfrac{1}{4}\ln 5$

$$-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

$$8C(5,8)=4C(1,4)-\frac{\pi}{\sqrt{2}}+\ln 5$$

$$-\sqrt{2}\ln(\sqrt{2}+1)$$

$$=C(1,1)-\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 10$$

$$-\sqrt{2}\ln(\sqrt{2}+1).$$

于定理5令 $a=7$，$k=4$ 即得

$$2C(7,8)-C(7,4)$$

$$=A(7,4)=-\frac{\pi}{4\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{3}.$$

于命题2令 $a=3$，$b=1$，$k=4$ 即得

$$C(7,4)=C(3,4)+\frac{1}{4}\ln\frac{7}{3}-\frac{1}{3}.$$

于是 $$2C(7,8)=C(3,4)-\frac{\pi}{4\sqrt{2}}+\frac{1}{4}\ln\frac{7}{3}+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\begin{aligned}8C(7,8)&=4C(3,4)-\frac{\pi}{\sqrt{2}}+\ln\frac{7}{3}+\sqrt{2}\ln(\sqrt{2}+1)\\&=C(1,1)-\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 14+\sqrt{2}\ln(\sqrt{2}+1).\end{aligned}$$

于定理5令 $a=1$，$k=6$ 即得

$$\begin{aligned}2C(1,12)-C(1,6)&=A(1,6)\\&=\frac{\pi}{6}+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}),\end{aligned}$$

于是 $$\begin{aligned}12C(1,12)&=6C(1,6)+\pi+\sqrt{3}\ln(2+\sqrt{3})\\&=C(1,1)+\frac{\pi\sqrt{3}}{2}+\pi+\frac{1}{2}\ln 12+\sqrt{3}\ln(2+\sqrt{3}).\end{aligned}$$

于定理5令 $a=5$，$k=6$ 即得

$$\begin{aligned}2C(5,12)-C(5,6)&=A(5,6)\\&=\frac{\pi}{6}-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}),\end{aligned}$$

于是 $$\begin{aligned}12C(5,12)&=6C(5,6)+\pi-\sqrt{3}\ln(2+\sqrt{3})\\&=C(1,1)-\frac{\pi\sqrt{3}}{2}+\pi+\frac{1}{2}\ln 300\end{aligned}$$

$$-\sqrt{3}\ln(2+\sqrt{3}).$$

于定理5令$a=7$，$k=6$即得

$$2C(7,12)-C(7,6)=A(7,6)$$

$$=-\frac{\pi}{6}-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3})+1.$$

于命题2令$a=1$，$b=1$，$k=6$即得

$$C(7,6)=C(1,6)+\frac{1}{6}\ln 7-1.$$

于是
$$2C(7,12)=C(1,6)-\frac{\pi}{6}+\frac{1}{6}\ln 7$$

$$-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}),$$

$$12C(7,12)=6C(1,6)-\pi+\ln 7-\sqrt{3}\ln(2+\sqrt{3})$$

$$=C(1,1)+\frac{\pi\sqrt{3}}{2}-\pi+\frac{1}{2}\ln 7^2\cdot 12$$

$$-\sqrt{3}\ln(2+\sqrt{3}).$$

于定理5令$a=11$，$k=6$即得

$$2C(11,12)-C(11,6)$$

$$=A(11,6)=-\frac{\pi}{6}+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{5}.$$

于命题2令$a=5$，$b=1$，$k=6$即得

$$C(11,6)=C(5,6)+\frac{1}{6}\ln\frac{11}{5}-\frac{1}{5}.$$

于是
$$2C(11,12)=C(5,6)-\frac{\pi}{6}+\frac{1}{6}\ln\frac{11}{5}$$

$$+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}),$$

$$12C(11,12)=6C(5,6)-\pi+\ln\frac{11}{5}$$

$$+\sqrt{3}\ln(2+\sqrt{3})$$

$$=C(1,1)-\frac{\pi\sqrt{3}}{2}-\pi+\frac{1}{2}\ln 12\cdot 11^2$$

$$+\sqrt{3}\ln(2+\sqrt{3}).$$

应用积分方法还可求得

$$A(1,5)=\frac{\pi}{25}\sqrt{\frac{5+\sqrt{5}}{2}}+\frac{2\pi}{25}\sqrt{\frac{5-\sqrt{5}}{2}}+\frac{1}{5}\ln 2$$

$$+\frac{1}{2\sqrt{5}}\ln\frac{3+\sqrt{5}}{2}$$

$$=0.8883135726\cdots;$$

$$A(2,5)=\frac{\pi}{5}\left(\frac{\sqrt{5}-1}{\sqrt{50+10\sqrt{5}}}-\frac{1}{5}\sqrt{\frac{5+\sqrt{5}}{2}}\right)$$

$$+\frac{2\pi}{5}\left(\frac{\sqrt{5}+1}{\sqrt{50-10\sqrt{5}}}-\frac{1}{5}\sqrt{\frac{5-\sqrt{5}}{2}}\right)$$

$$-\frac{1}{5}\ln 2+\frac{1}{2\sqrt{5}}\ln\frac{3+\sqrt{5}}{2}$$

$$=0.4069016342\cdots;$$

$$A(3,5)=\frac{\pi}{5}\left(\frac{1}{5}\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+3}{\sqrt{50+10\sqrt{5}}}\right)$$

$$+\frac{2\pi}{5}\left(\frac{1}{5}\sqrt{\frac{5-\sqrt{5}}{2}}+\frac{3-\sqrt{5}}{\sqrt{50-10\sqrt{5}}}\right)$$

$$+\frac{1}{5}\ln 2-\frac{1}{2\sqrt{5}}\ln\frac{3+\sqrt{5}}{2}$$

$$=0.2537515655\cdots;$$

$$A(4,5)=\frac{\pi}{5}\left(\frac{2\sqrt{5}+2}{\sqrt{50+10\sqrt{5}}}-\frac{1}{5}\sqrt{\frac{5+\sqrt{5}}{2}}\right)$$

$$+\frac{2\pi}{5}\left(\frac{2\sqrt{5}-2}{\sqrt{50-10\sqrt{5}}}-\frac{1}{5}\sqrt{\frac{5-\sqrt{5}}{2}}\right)$$

$$-\frac{1}{5}\ln 2-\frac{1}{2\sqrt{5}}\ln\frac{3+\sqrt{5}}{2}$$

$$=0.1806457593\cdots;$$

$$A(6,5)=-A(1,5)+1;\quad A(7,5)=-A(2,5)+\frac{1}{2};$$

$$A(8,5)=-A(3,5)+\frac{1}{3};$$

$$A(9,5)=-A(4,5)+\frac{1}{4}.$$

$$A(1,8)=\frac{\sqrt{2}-1}{8\sqrt{4-2\sqrt{2}}}\ln\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$+\frac{\sqrt{2}+1}{8\sqrt{4+2\sqrt{2}}}\ln\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}$$

$$+\frac{\pi}{2}\left(\frac{1}{4\sqrt{4-2\sqrt{2}}}+\frac{1}{4\sqrt{4+2\sqrt{2}}}\right)$$

$$=0.9246517057\cdots;$$

$$A(3,8)=-\frac{1}{8\sqrt{4-2\sqrt{2}}}\ln\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$+\frac{1}{8\sqrt{4+2\sqrt{2}}}\ln\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}$$

$$+\frac{\pi}{2}\left(\frac{1}{4\sqrt{4-2\sqrt{2}}}-\frac{1}{4\sqrt{4+2\sqrt{2}}}\right)$$

$$=0.2738982192\cdots;$$

$$A(5,8)=\frac{1}{8\sqrt{4-2\sqrt{2}}}\ln\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$-\frac{1}{8\sqrt{4+2\sqrt{2}}}\ln\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}$$

$$+\frac{\pi}{2}\left(\frac{1}{4\sqrt{4-2\sqrt{2}}}-\frac{1}{4\sqrt{4+2\sqrt{2}}}\right)$$

$$=0.1511562038\cdots;$$

$$A(7,8)=-\frac{\sqrt{2}-1}{8\sqrt{4-2\sqrt{2}}}\ln\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$-\frac{\sqrt{2}+1}{8\sqrt{4+2\sqrt{2}}}\ln\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}$$

$$+\frac{\pi}{2}\left(\frac{\sqrt{2}+1}{4\sqrt{4+2\sqrt{2}}}+\frac{\sqrt{2}-1}{4\sqrt{4-2\sqrt{2}}}\right)$$

$$=0.1015204471\cdots;$$

$$A(9,8)=-A(1,8)+1;$$

$$A(11,8)=-A(3,8)+\frac{1}{3};$$

$$A(13,8)=-A(5,8)+\frac{1}{5};$$

$$A(15,8)=-A(7,8)+\frac{1}{7}.$$

定理7 $5C(1,5)=C(1,1)+\frac{5}{4}A(1,5)+\frac{5}{2}A(2,5)+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20;$

$$5C(2,5)=C(1,1)+\frac{5}{4}A(1,5)-\frac{5}{2}A(2,5)+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20;$$

$$5C(3,5)=C(1,5)-\frac{15}{4}A(1,5)+\frac{5}{2}A(2,5)+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20+\ln 6;$$

$$5C(4,5)=C(1,1)+\frac{5}{4}A(1,5)-\frac{5}{2}A(2,5)-\frac{15}{4}A(4,5)+\frac{1}{4}\ln 20;$$

$$10C(1,10)=C(1,1)+\frac{25}{4}A(1,5)+\frac{5}{2}A(2,5)+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20;$$

$$10C(3,10)=C(1,1)-\frac{15}{4}A(1,5)+\frac{5}{2}A(2,5)$$

$$+5A(3,5)+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20+\ln 6;$$

$$10C(7,10)=C(1,1)+\frac{5}{4}A(1,5)-\frac{15}{2}A(2,5)$$

$$+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20+\ln\frac{7}{2};$$

$$10C(9,10)=C(1,1)+\frac{5}{4}A(1,5)-\frac{5}{2}A(2,5)$$

$$-\frac{35}{4}A(4,5)+\frac{1}{4}\ln 20+\ln\frac{9}{4}.$$

证明 于(1) 令 $a=1, k=5$，则有

$$C(1,10)-\frac{1}{2}C(3,5)+\frac{1}{10}\ln 6=A(1,5),$$

$$2C(1,10)-C(3,5)+\frac{1}{5}\ln 6=2A(1,5).$$

于定理5令 $a=1$，$k=5$，则有

$$2C(1,10)-C(1,5)=A(1,5).$$

由此得出

$$C(1,5)-C(3,5)+\frac{1}{5}\ln 6=A(1,5).$$

于定理5令 $a=2$，$k=5$，则有

$$C(1,5)-C(2,5)=A(2,5).$$

于定理5令 $a=4$，$k=5$，则有

$$C(2,5)-C(4,5)=A(4,5).$$

于定理4令 $a=1$，$k=1$，$m=5$，则有

$$C(1,5)+C(2,5)+C(3,5)+C(4,5)$$

$$=\frac{4}{5}C(1,1)+\frac{1}{5}\ln 120.$$

由以上四个等式即可解出 $C(1,5)$、$C(2,5)$、$C(3,5)$ 以及 $C(4,5)$。

于定理 5 令 $a=1$，$k=5$，则有

$$2C(1,10)=C(1,5)+A(1,5),$$

于是
$$\begin{aligned}10C(1,10)&=5C(1,5)+5A(1,5)\\&=C(1,1)+\frac{25}{4}A(1,5)+\frac{5}{2}A(2,5)\\&\quad+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20.\end{aligned}$$

于定理 5 令 $a=3$，$k=5$，则有

$$2C(3,10)=C(3,5)+A(3,5),$$

于是
$$\begin{aligned}10C(3,10)&=5C(3,5)+5A(3,5)\\&=C(1,1)-\frac{15}{4}A(1,5)+\frac{5}{2}A(2,5)\\&\quad+5A(3,5)+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20\\&\quad+\ln 6.\end{aligned}$$

于定理 5 令 $a=7$，$k=5$，则有

$$\begin{aligned}2C(7,10)&=C(7,5)+A(7,5)\\&=C(7,5)-A(2,5)+\frac{1}{2}.\end{aligned}$$

于命题 2 令 $a=2$，$b=1$，$k=5$，则有

$$C(7,5)=C(2,5)+\frac{1}{5}\ln\frac{7}{2}-\frac{1}{2}.$$

于是
$$2C(7,10)=C(2,5)-A(2,5)+\frac{1}{5}\ln\frac{7}{2},$$

$$10C(7,10)=5C(2,5)-5A(2,5)+\ln\frac{7}{2}$$

$$=C(1,1)+\frac{5}{4}A(1,5)-\frac{15}{2}A(2,5)$$

$$+\frac{5}{4}A(4,5)+\frac{1}{4}\ln 20+\ln\frac{7}{2}.$$

于定理5令 $a=9$，$k=5$，则有

$$2C(9,10)=C(9,5)+A(9,5)$$

$$=C(9,5)-A(4,5)+\frac{1}{4}.$$

于命题2令 $a=4$，$b=1$，$k=5$，则有

$$C(9,5)=C(4,5)+\frac{1}{5}\ln\frac{9}{4}-\frac{1}{4}.$$

于是 $2C(9,10)=C(4,5)-A(4,5)+\frac{1}{5}\ln\frac{9}{4}$,

$$10C(9,10)=5C(4,5)-5A(4,5)+\ln\frac{9}{4}$$

$$=C(1,1)+\frac{5}{4}A(1,5)-\frac{5}{2}A(2,5)$$

$$-\frac{35}{4}A(4,5)+\frac{1}{4}\ln 20+\frac{9}{4}.$$

定理8 $16C(1,16)=C(1,1)+\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 2$

$$+\sqrt{2}\ln(\sqrt{2}+1)+8A(1,8),$$

$$16C(3,16)=C(1,1)+\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 6$$

$$-\sqrt{2}\ln(\sqrt{2}+1)+8A(3,8),$$

$$16C(5,16)=C(1,1)-\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 10$$

$$-\sqrt{2}\ln(\sqrt{2}+1)+8A(5,8),$$

$$16C(7,16)=C(1,1)-\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 14$$

$$+\sqrt{2}\ln(\sqrt{2}+1)+8A(7,8),$$

$$16C(9,16)=C(1,1)+\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 18$$

$$+\sqrt{2}\ln(\sqrt{2}+1)-8A(1,8),$$

$$16C(11,16)=C(1,1)+\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 22$$

$$-\sqrt{2}\ln(\sqrt{2}+1)-8A(3,8),$$

$$16C(13,16)=C(1,1)-\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 26$$

$$-\sqrt{2}\ln(\sqrt{2}+1)-8A(5,8),$$

$$16C(15,16)=C(1,1)-\frac{\pi}{\sqrt{2}}-\frac{\pi}{2}+\ln 30$$

$$+\sqrt{2}\ln(\sqrt{2}+1)-8A(7,8).$$

证明 于定理5 令 $a=1$, $k=8$, 则有

$$2C(1,16)-C(1,8)=A(1,8).$$

于是
$$\begin{aligned}16C(1,16)&=8C(1,8)+8A(1,8)\\&=C(1,1)+\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 2\\&\quad+\sqrt{2}\ln(\sqrt{2}+1)+8A(1,8).\end{aligned}$$

仿此可得 $16C(3,16)$、$16C(5,16)$ 以及 $16C(7,16)$。

于定理5令 $a=9$, $k=8$, 则有

$$\begin{aligned}2C(9,16)&=C(9,8)+A(9,8)\\&=C(9,8)-A(1,8)+1.\end{aligned}$$

于命题2令 $a=1$，$b=1$，$k=8$，则有

$$C(9,8)=C(1,8)+\frac{1}{8}\ln 9-1.$$

于是 $2C(9,16)=C(1,8)+\frac{1}{8}\ln 9-A(1,8)$,

$$16C(9,16)=8C(1,8)+\ln 9-8A(1,8)$$
$$=C(1,1)+\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}+\ln 18$$
$$+\sqrt{2}\ln(\sqrt{2}+1)-8A(1,8).$$

仿此可得 $16C(11,16)$、$16C(13,16)$ 以及 $16C(15,16)$.

应用定理6、7、8所给出的各个常数 $C(a,k)$，由以下的诸定理将推得各种类型的无穷级数之和.

三、无穷级数的求和

1. $\sum_{n=1}^{\infty}\left[\frac{1}{k(n-1)+a}-\frac{1}{k(n-1)+b}\right]$

定理9 设 a、b、k 是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(n-1)+a}-\frac{1}{k(n-1)+b}\right]$$

$$=C(a,k)-C(b,k)+\frac{1}{k}\ln\frac{b}{a}.$$

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{1}{k(i-1)+a}-\frac{1}{k(i-1)+b}\right]$$

$$=C(a,k)-C(b,k)+\frac{1}{k}\ln\frac{nk-k+a}{nk-k+b}$$

$$+\frac{1}{k}\ln\frac{b}{a}+\varepsilon_n(a,k)-\varepsilon_n(b,k).$$

令 $n\to\infty$并取极限，定理即得证。

例1 求和 $\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-2)(4n-1)4n(4n+1)}$.

解 于定理9令 $a=1$，$b=2$，$k=4$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-2)}=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-2}\right)$$

$$=C(1,4)-\frac{1}{2}C(1,2)+\frac{1}{4}\ln 2.$$

根据定理6，有

$$C(1,4)-\frac{1}{2}C(1,2)=\frac{\pi}{8}.$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-2)}=\frac{\pi}{8}+\frac{1}{4}\ln 2.$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-2)(4n-1)}=\frac{\pi}{8}-\frac{1}{4}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)4n}=-\frac{\pi}{8}+\frac{3}{4}\ln 2.$$

于定理9令 $a=4$，$b=5$，$k=4$，则有

$$\sum_{n=1}^{\infty}\frac{1}{4n(4n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{1}{4n+1}\right)$$

$$=\frac{1}{4}C(1,1)-C(5,4)+\frac{1}{4}\ln\frac{5}{2^2}.$$

于命题2令 $a=1$，$b=1$，$k=4$，则有

$$C(5,4)=C(1,4)+\frac{1}{4}\ln 5-1.$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n(4n+1)}=\frac{1}{4}C(1,1)-C(1,4)-\frac{1}{2}\ln 2+1.$

根据定理6，有

$$\frac{1}{4}C(1,1)-C(1,4)=-\frac{\pi}{8}-\frac{1}{4}\ln 2.$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n(4n+1)}=-\frac{\pi}{8}-\frac{3}{4}\ln 2+1.$

从上面所得的结果即可顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-2)(4n-1)}$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(4n-2)}-\frac{1}{(4n-2)(4n-1)}\right]$$
$$=\frac{1}{4}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-2)(4n-1)4n}$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-2)(4n-1)}-\frac{1}{(4n-1)4n}\right]$$
$$=\frac{\pi}{8}-\frac{1}{2}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)4n(4n+1)}$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)4n}-\frac{1}{4n(4n+1)}\right]$$
$$=\frac{3}{4}\ln 2-\frac{1}{2},$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-2)(4n-1)4n}$$
$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(4n-2)(4n-1)}-\frac{1}{(4n-2)(4n-1)4n}\right]$$
$$=-\frac{\pi}{24}+\frac{1}{4}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-2)(4n-1)4n(4n+1)}$$
$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-2)(4n-1)4n}-\frac{1}{(4n-1)4n(4n+1)}\right]$$

$$=\frac{\pi}{24}-\frac{5}{12}\ln 2+\frac{1}{6}.$$

于是

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-2)(4n-1)4n(4n+1)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(4n-2)(4n-1)4n}-\frac{1}{(4n-2)(4n-1)4n(4n+1)}\right]$$

$$=-\frac{\pi}{48}+\frac{1}{6}\ln 2-\frac{1}{24}.$$

例2 求和 $\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)(8n-1)}$.

解 于定理9令 $a=1$, $b=4$, $k=8$, 即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)}=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-4}\right)$$

$$=\frac{\pi}{24\sqrt{2}}+\frac{\pi}{48}+\frac{1}{12}\ln 2+\frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1).$$

令 $a=4$, $b=7$, $k=8$, 即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(8n-4)(8n-1)}=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{8n-4}-\frac{1}{8n-1}\right)$$

$$=\frac{\pi}{24\sqrt{2}}+\frac{\pi}{48}-\frac{1}{12}\ln 2-\frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1).$$

于是

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)(8n-1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-4)}-\frac{1}{(8n-4)(8n-1)}\right]$$

$$=\frac{1}{36}\ln 2+\frac{1}{36\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此求得一系列的级数之和：

(一) $$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)}$$

$$=\frac{\pi}{24\sqrt{3}}+\frac{1}{3}\ln 2-\frac{1}{8}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(6n-3)(6n-2)}$$

$$=\frac{1}{4}\ln 3-\frac{1}{3}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-3)(6n-2)(6n-1)}$$

$$=-\frac{\pi}{24\sqrt{3}}+\frac{1}{3}\ln 2-\frac{1}{8}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)6n}$$

$$=\frac{5\pi}{24\sqrt{3}}-\frac{1}{3}\ln 2-\frac{1}{8}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)6n(6n+1)}$$

$$=\frac{1}{4}\ln 3+\frac{1}{3}\ln 2-\frac{1}{2}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)3n}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)}+\frac{1}{(6n-2)(6n-1)6n}\right]$$

$$=\frac{\pi}{4\sqrt{3}}-\frac{1}{4}\ln 3,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)3n}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)}-\frac{1}{(6n-2)(6n-1)6n}\right]$$

$$=-\frac{\pi}{6\sqrt{3}}+\frac{2}{3}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)}\right.$$

$$\left.-\frac{1}{(6n-4)(6n-3)(6n-2)}\right]$$

$$=\frac{\pi}{72\sqrt{3}}-\frac{1}{8}\ln 3+\frac{2}{9}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\frac{\pi}{72\sqrt{3}}+\frac{1}{8}\ln 3-\frac{2}{9}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-3)(6n-2)(6n-1)6n}$$

$$=-\frac{\pi}{12\sqrt{3}}+\frac{2}{9}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)6n(6n+1)}$$

$$=\frac{5\pi}{72\sqrt{3}}-\frac{2}{9}\ln 2-\frac{1}{8}\ln 3+\frac{1}{6}.$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)3n(3n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)}\right.$$

$$\left.+\frac{1}{(6n-2)(6n-1)6n(6n+1)}\right]$$

$$=\frac{\pi}{12\sqrt{3}}-\frac{1}{4}\ln 3+\frac{1}{6},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)3n(3n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)}\right.$$

$$\left.-\frac{1}{(6n-2)(6n-1)6n(6n+1)}\right]$$

$$=-\frac{\pi}{18\sqrt{3}}+\frac{4}{9}\ln 2-\frac{1}{6},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)}\right.$$

$$\left.-\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)}\right]$$

$$=\frac{1}{9}\ln 2-\frac{1}{16}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n}$$

$$=\frac{7\pi}{288\sqrt{3}}+\frac{1}{32}\ln 3-\frac{1}{9}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-3)(6n-2)(6n-1)6n(6n+1)}$$

$$=-\frac{11\pi}{288\sqrt{3}}+\frac{1}{9}\ln 2+\frac{1}{32}\ln 3-\frac{1}{24},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)6n}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}\right.$$

$$\left.-\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n}\right]$$

$$=-\frac{7\pi}{1440\sqrt{3}}+\frac{2}{45}\ln 2-\frac{3}{160}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n(6n+1)}$$

$$=\frac{\pi}{80\sqrt{3}}-\frac{2}{45}\ln 2+\frac{1}{120},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)6n(6n+1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)6n}\right.$$

$$\left.-\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n(6n+1)}\right]$$

$$=-\frac{5\pi}{1728\sqrt{3}}+\frac{2}{135}\ln 2-\frac{1}{320}\ln 3-\frac{1}{720}.$$

(二) $$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)}$$

$$=\frac{\pi}{8\sqrt{2}}-\frac{\pi}{16}+\frac{1}{8}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-6)(8n-5)(8n-4)}$$

$$=-\frac{\pi}{8\sqrt{2}}+\frac{3\pi}{32}-\frac{3}{16}\ln 2+\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{4}\ln 2-\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-4)(8n-3)(8n-2)}$$

$$=\frac{\pi}{8\sqrt{2}}-\frac{3\pi}{32}-\frac{3}{16}\ln 2+\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-2)(8n-1)}$$

$$=-\frac{\pi}{8\sqrt{2}}+\frac{\pi}{16}+\frac{1}{8}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-2)(8n-1)8n}$$

$$=\frac{\pi}{8\sqrt{2}}+\frac{\pi}{32}-\frac{5}{16}\ln 2-\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-3)(4n-2)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-6)(8n-5)}-\frac{1}{(8n-3)(8n-2)(8n-1)}\right]$$

$$=\frac{\pi}{4\sqrt{2}}-\frac{\pi}{8},$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-6)(8n-5)}\right.$$

$$\left.-\frac{1}{(8n-6)(8n-5)(8n-4)}\right]$$

$$=\frac{\pi}{12\sqrt{2}}-\frac{5\pi}{96}+\frac{5}{48}\ln 2-\frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=-\frac{\pi}{24\sqrt{2}}+\frac{\pi}{32}-\frac{7}{48}\ln 2+\frac{1}{6\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=-\frac{\pi}{24\sqrt{2}}+\frac{\pi}{32}+\frac{7}{48}\ln 2-\frac{1}{6\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{\pi}{12\sqrt{2}}-\frac{5\pi}{96}-\frac{5}{48}\ln 2+\frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-2)(8n-1)8n}$$

$$=-\frac{\pi}{12\sqrt{2}}+\frac{\pi}{96}+\frac{7}{48}\ln 2+\frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-3)(4n-2)(4n-1)4n}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)}\right.$$

$$-\frac{1}{(8n-3)(8n-2)(8n-1)8n}\Bigg]$$

$$=\frac{\pi}{6\sqrt{2}}-\frac{\pi}{16}-\frac{1}{24}\ln 2-\frac{1}{6\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)}$$

$$-\frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)}\Bigg]$$

$$=\frac{\pi}{32\sqrt{2}}-\frac{\pi}{48}+\frac{1}{16}\ln 2-\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=-\frac{7}{96}\ln 2+\frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=-\frac{\pi}{32\sqrt{2}}+\frac{\pi}{48}+\frac{1}{16}\ln 2-\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-4)(8n-3)(8n-2)(8n-1)8n}$$

$$=\frac{\pi}{24\sqrt{2}}-\frac{\pi}{64}-\frac{1}{16}\ln 2;$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$-\frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}\Bigg]$$

$$=\frac{\pi}{160\sqrt{2}}-\frac{\pi}{240}+\frac{13}{480}\ln 2-\frac{7}{240\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$=\frac{\pi}{160\sqrt{2}}-\frac{\pi}{240}-\frac{13}{480}\ln 2+\frac{7}{240\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(8n-4)(8n-3)\cdots 8n}$$

$$=-\frac{7\pi}{480\sqrt{2}}+\frac{7\pi}{960}+\frac{1}{40}\ln 2-\frac{1}{80\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}$$

$$=\frac{13}{1440}\ln 2-\frac{7}{720\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-6)(8n-5)(8n-4)\cdots 8n}$$

$$=\frac{\pi}{288\sqrt{2}}-\frac{11\pi}{5760}-\frac{5}{576}\ln 2$$

$$+\frac{1}{144\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)\cdots 8n}$$

$$=-\frac{\pi}{2016\sqrt{2}}+\frac{11\pi}{40320}+\frac{17}{6720}\ln 2$$

$$-\frac{1}{420\sqrt{2}}\ln(\sqrt{2}+1).$$

(三) $$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(8n-3)}$$

$$=-\frac{\pi}{32\sqrt{2}}+\frac{\pi}{32}+\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(8n-3)(8n-1)}$$

$$=\frac{\pi}{32\sqrt{2}}-\frac{\pi}{32}+\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-1)(8n+1)}$$

$$=\frac{\pi}{32\sqrt{2}}+\frac{\pi}{32}-\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{8}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-3)(4n-1)(4n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(8n-3)}\right.$$

$$\left.-\frac{1}{(8n-3)(8n-1)(8n+1)}\right]$$

$$=-\frac{\pi}{16\sqrt{2}}+\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{8},$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(8n-3)(8n-1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(8n-3)}\right.$$

$$\left.-\frac{1}{(8n-5)(8n-3)(8n-1)}\right]$$

$$= -\frac{\pi}{96\sqrt{2}} + \frac{\pi}{96},$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$= -\frac{\pi}{96} + \frac{1}{48\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{48};$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$= \frac{1}{8} \sum_{n=1}^{\infty} \left[\frac{1}{(8n-7)(8n-5)(8n-3)(8n-1)} \right.$$

$$\left. - \frac{1}{(8n-5)(8n-3)(8n-1)(8n+1)} \right]$$

$$= -\frac{\pi}{768\sqrt{2}} + \frac{\pi}{384} - \frac{1}{384\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{384}.$$

（四）$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)}$$

$$= -\frac{\pi}{16\sqrt{3}} + \frac{\pi}{16} + \frac{1}{12} \ln 2 - \frac{1}{16} \ln 3$$

$$+ \frac{1}{8\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)}$$

$$= \frac{\pi}{12\sqrt{3}} - \frac{\pi}{24} + \frac{1}{8} \ln 3 - \frac{1}{6} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)}$$

$$= -\frac{5\pi}{48\sqrt{3}} + \frac{\pi}{16} + \frac{1}{4} \ln 2 - \frac{1}{16} \ln 3$$

$$-\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-7)(12n-6)}$$

$$=\frac{7\pi}{48\sqrt{3}}-\frac{\pi}{12}-\frac{1}{16}\ln 3-\frac{1}{6}\ln 2$$

$$+\frac{1}{4\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-6)(12n-5)}$$

$$=\frac{1}{8}\ln 3+\frac{1}{12}\ln 2-\frac{1}{4\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-6)(12n-5)(12n-4)}$$

$$=-\frac{7\pi}{48\sqrt{3}}+\frac{\pi}{12}-\frac{1}{16}\ln 3-\frac{1}{6}\ln 2$$

$$+\frac{1}{4\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)}$$

$$=\frac{5\pi}{48\sqrt{3}}-\frac{\pi}{16}-\frac{1}{16}\ln 3+\frac{1}{4}\ln 2$$

$$-\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-4)(12n-3)(12n-2)}$$

$$=-\frac{\pi}{12\sqrt{3}}+\frac{\pi}{24}+\frac{1}{8}\ln 3-\frac{1}{6}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-3)(12n-2)(12n-1)}$$

$$= \frac{\pi}{16\sqrt{3}} - \frac{\pi}{16} + \frac{1}{12}\ln 2 - \frac{1}{16}\ln 3$$

$$+ \frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-2)(12n-1)12n}$$

$$= \frac{\pi}{16\sqrt{3}} + \frac{\pi}{12} - \frac{1}{6}\ln 2 - \frac{1}{16}\ln 3$$

$$- \frac{1}{4\sqrt{3}}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)}\right.$$

$$\left. - \frac{1}{(12n-5)(12n-4)(12n-3)}\right]$$

$$= -\frac{\pi}{6\sqrt{3}} + \frac{\pi}{8} - \frac{1}{6}\ln 2 + \frac{1}{4\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$= \frac{1}{3}\left[\frac{1}{(12n-11)(12n-10)(12n-9)}\right.$$

$$\left. - \frac{1}{(12n-10)(12n-9)(12n-8)}\right]$$

$$= -\frac{7\pi}{144\sqrt{3}} + \frac{5\pi}{144} + \frac{1}{12}\ln 2 - \frac{1}{16}\ln 3$$

$$+\frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$=\frac{\pi}{16\sqrt{3}}-\frac{5\pi}{144}+\frac{1}{16}\ln 3-\frac{5}{36}\ln 2$$

$$+\frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$=-\frac{\pi}{12\sqrt{3}}+\frac{7\pi}{144}+\frac{5}{36}\ln 2-\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$=\frac{7\pi}{144\sqrt{3}}-\frac{\pi}{36}-\frac{1}{16}\ln 3-\frac{1}{12}\ln 2$$

$$+\frac{1}{6\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$=\frac{7\pi}{144\sqrt{3}}-\frac{\pi}{36}+\frac{1}{16}\ln 3+\frac{1}{12}\ln 2$$

$$-\frac{1}{6\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$=-\frac{\pi}{12\sqrt{3}}+\frac{7\pi}{144}-\frac{5}{36}\ln 2+\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$
=\frac{\pi}{16\sqrt{3}}-\frac{5\pi}{144}-\frac{1}{16}\ln 3+\frac{5}{36}\ln 2
$$

$$
-\frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),
$$

$$
\sum_{n=1}^{\infty}\frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)}
$$

$$
=-\frac{7\pi}{144\sqrt{3}}+\frac{5\pi}{144}+\frac{1}{16}\ln 3-\frac{1}{12}\ln 2
$$

$$
-\frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),
$$

$$
\sum_{n=1}^{\infty}\frac{1}{(12n-3)(12n-2)(12n-1)12n}
$$

$$
=-\frac{7\pi}{144}+\frac{1}{12}\ln 2+\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}).
$$

$$
\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)}
$$

$$
=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}\right.
$$

$$
\left.-\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)}\right]
$$

$$
=-\frac{\pi}{9\sqrt{3}}+\frac{5\pi}{72}-\frac{1}{18}\ln 2+\frac{1}{12\sqrt{3}}\ln(2+\sqrt{3}),
$$

$$
\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}
$$

$$
=\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}\right.
$$

$$
\left.-\frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)}\right]
$$

$$= -\frac{1}{36\sqrt{3}} + \frac{5\pi}{288} - \frac{1}{32}\ln 3 + \frac{1}{18}\ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= \frac{7\pi}{192\sqrt{3}} - \frac{\pi}{48} + \frac{1}{64}\ln 3 - \frac{5}{72}\ln 2$$

$$+ \frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= -\frac{19\pi}{576\sqrt{3}} + \frac{11\pi}{576} + \frac{1}{64}\ln 3 + \frac{1}{18}\ln 2$$

$$- \frac{7}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= -\frac{1}{32}\ln 3 - \frac{1}{24}\ln 2 + \frac{1}{12\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$= \frac{19\pi}{576\sqrt{3}} - \frac{11\pi}{576} + \frac{1}{64}\ln 3 + \frac{1}{18}\ln 2$$

$$- \frac{7}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-6)(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= -\frac{7\pi}{192\sqrt{3}} + \frac{\pi}{48} + \frac{1}{64}\ln 3 - \frac{5}{72}\ln 2$$

$$+ \frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$=\frac{\pi}{36\sqrt{3}}-\frac{5\pi}{288}-\frac{1}{32}\ln 3+\frac{1}{18}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)12n}$$

$$=-\frac{7\pi}{576\sqrt{3}}+\frac{\pi}{48}+\frac{1}{64}\ln 3-\frac{1}{24}\ln 2$$

$$-\frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}\right]$$

$$=-\frac{\pi}{18\sqrt{3}}+\frac{5\pi}{144},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12-7)}\right.$$

$$\left.-\frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}\right]$$

$$=-\frac{37\pi}{2880\sqrt{3}}+\frac{11\pi}{1440}-\frac{3}{320}\ln 3+\frac{1}{40}\ln 2$$

$$-\frac{1}{120\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}$$

$$=\frac{\pi}{72\sqrt{3}}-\frac{23\pi}{2880}-\frac{1}{40}\ln 2+\frac{11}{480\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-4)}$$

$$=-\frac{19\pi}{2880\sqrt{3}}+\frac{11\pi}{2880}+\frac{3}{320}\ln 3+\frac{7}{360}\ln 2$$

$$-\frac{1}{32\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-7)(12n-6)\cdots(12n-3)}$$

$$=-\frac{19\pi}{2880\sqrt{3}}+\frac{11\pi}{2880}-\frac{3}{320}\ln 3-\frac{7}{360}\ln 2$$

$$+\frac{1}{32\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-6)(12n-5)\cdots(12n-2)}$$

$$=\frac{\pi}{72\sqrt{3}}-\frac{23\pi}{2880}+\frac{1}{40}\ln 2-\frac{11}{480\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-6)(12n-5)(12n-4)\cdots(12n-1)}$$

$$=-\frac{37\pi}{2880\sqrt{3}}+\frac{11\pi}{1440}+\frac{3}{320}\ln 3-\frac{1}{40}\ln 2$$

$$+\frac{1}{120\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)\cdots 12n}$$

$$=\frac{23\pi}{2880\sqrt{3}}-\frac{11\pi}{1440}-\frac{3}{320}\ln 3+\frac{7}{360}\ln 2$$

$$+\frac{1}{120\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)\cdots 6n}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-4)(12n-3)\cdots 12n}\right]$$

$$=-\frac{\pi}{48\sqrt{3}}+\frac{11\pi}{720}+\frac{1}{180}\ln 2-\frac{1}{60\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}$$

$$=\frac{1}{6}\sum_{n=1}^{8}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}\right.$$

$$\left.-\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}\right]$$

$$=-\frac{77\pi}{17280\sqrt{3}}+\frac{\pi}{384}-\frac{1}{640}\ln 3+\frac{1}{120}\ln 2$$

$$-\frac{1}{192\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)}$$

$$=\frac{59\pi}{17280\sqrt{3}}-\frac{17\pi}{8640}-\frac{1}{640}\ln 3-\frac{1}{135}\ln 2$$

$$+\frac{13}{1440\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-3)}$$

$$=\frac{1}{320}\ln 3+\frac{7}{1080}\ln 2-\frac{1}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-7)(12n-6)\cdots(12n-2)}$$

$$=-\frac{59\pi}{17280\sqrt{3}}+\frac{17\pi}{8640}-\frac{1}{640}\ln 3-\frac{1}{135}\ln 2$$

$$+\frac{13}{1440\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-6)(12n-5)\cdots(12n-1)}$$

$$=\frac{77\pi}{17280\sqrt{3}}-\frac{\pi}{384}+\frac{1}{120}\ln 2-\frac{1}{640}\ln 3$$

$$-\frac{1}{192\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)}$$

$$=\frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)}\right]$$

$$=-\frac{17\pi}{15120\sqrt{3}}+\frac{79\pi}{120960}+\frac{17}{7560}\ln 2$$

$$-\frac{41}{20160\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)}$$

$$=\frac{59\pi}{120960\sqrt{3}}-\frac{17\pi}{60480}-\frac{3}{4480}\ln 3$$

$$-\frac{1}{504}\ln 2+\frac{1}{360\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-2)}$$

$$=\frac{59\pi}{120960\sqrt{3}}-\frac{17\pi}{60480}+\frac{3}{4480}\ln 3$$

$$+\frac{1}{504}\ln 2-\frac{1}{306\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-7)(12n-6)\cdots(12n-1)}$$

$$=-\frac{17\pi}{15120\sqrt{3}}+\frac{79\pi}{120960}-\frac{17}{7560}\ln 2$$

$$+\frac{41}{20160\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)}\right.$$

$$\left.-\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)}\right]$$

$$=-\frac{13\pi}{64512\sqrt{3}}+\frac{113\pi}{967680}+\frac{1}{1890}\ln 2$$

$$+\frac{3}{35840}\ln 3-\frac{97}{161280\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}$$

$$=-\frac{3}{17920}\ln 3-\frac{1}{2016}\ln 2+\frac{1}{1440\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-1)}$$

$$=\frac{13\pi}{64512\sqrt{3}}-\frac{113\pi}{967680}+\frac{1}{1890}\ln 2$$

$$+\frac{3}{35840}\ln 3-\frac{97}{161280\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)}$$

$$=\frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}\right.$$

$$\left.-\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}\right]$$

$$=-\frac{13\pi}{580608\sqrt{3}}+\frac{113\pi}{8709120}+\frac{31}{272160}\ln 2$$

$$+\frac{1}{35840}\ln 3-\frac{209}{1451520\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)}$$

$$=-\frac{13\pi}{580608\sqrt{3}}+\frac{113\pi}{8709120}-\frac{31}{272160}\ln 2$$

$$-\frac{1}{35840}\ln 3+\frac{209}{1451520\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-1)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)}\right.$$

$$\left.-\frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)}\right]$$

$$=\frac{31}{1360800}\ln 2+\frac{1}{179200}\ln 3$$

$$-\frac{209}{7257600\sqrt{3}}\ln(2+\sqrt{3}).$$

（五）$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)}$

$$=\frac{\pi}{96}+\frac{1}{32}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-7)(12n-5)}$$

$$=\frac{\pi\sqrt{3}}{64}-\frac{5\pi}{192}-\frac{1}{64}\ln 3+\frac{1}{32\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-5)(12n-3)}$$

$$=-\frac{\pi\sqrt{3}}{64}+\frac{5\pi}{192}-\frac{1}{64}\ln 3+\frac{1}{32\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-3)(12n-1)}$$

$$=-\frac{\pi}{96}+\frac{1}{32}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-3)(12n-1)(12n+1)}$$

$$=\frac{\pi\sqrt{3}}{64}+\frac{5\pi}{192}-\frac{1}{64}\ln 3-\frac{1}{32\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{8}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-3)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)}\right.$$

$$\left.+\frac{1}{(12n-5)(12n-3)(12n-1)}\right]$$

$$=\frac{1}{16}\ln 3,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-3)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-3)(12n-1)}\right]$$

$$=\frac{\pi}{48},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)}\right.$$

$$\left.-\frac{1}{(12n-9)(12n-7)(12n-5)}\right]$$

$$=-\frac{\pi}{128\sqrt{3}}+\frac{7\pi}{1152}+\frac{1}{128}\ln 3$$

$$-\frac{1}{192\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$=\frac{\pi}{64\sqrt{3}}-\frac{5\pi}{576},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$=-\frac{\pi}{128\sqrt{3}}+\frac{7\pi}{1152}-\frac{1}{128}\ln 3$$

$$+\frac{1}{192\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$= -\frac{\pi}{128\sqrt{3}} - \frac{7\pi}{1152} + \frac{1}{128}\ln 3$$

$$+ \frac{1}{192\sqrt{3}}\ln(2+\sqrt{3}) + \frac{1}{48},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left. + \frac{1}{(12n-5)(12n-3)(12n-1)(12n+1)}\right]$$

$$= -\frac{\pi}{64\sqrt{3}} + \frac{1}{64}\ln 3 + \frac{1}{48},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left. - \frac{1}{(12n-5)(12n-3)(12n-1)(12n+1)}\right]$$

$$= \frac{7\pi}{576} - \frac{1}{96\sqrt{3}}\ln(2+\sqrt{3}) - \frac{1}{48},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$= \frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left. - \frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)}\right]$$

$$= -\frac{\pi\sqrt{3}}{1024} + \frac{17\pi}{9216} + \frac{1}{1024}\ln 3$$

$$-\frac{1}{1536\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$=\frac{\pi\sqrt{3}}{1024}-\frac{17\pi}{9216}+\frac{1}{1024}\ln 3-\frac{1}{1536\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-7)(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{7\pi}{4608}-\frac{1}{512}\ln 3-\frac{1}{384};$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n-3)}\right.$$

$$\left.-\frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{5120}+\frac{17\pi}{46080},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}$$

$$=\frac{\pi\sqrt{3}}{10240}-\frac{31\pi}{92160}+\frac{3}{10240}\ln 3$$

$$-\frac{1}{15360\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{3840};$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n+1)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}\right.$$

$$- \frac{1}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}\Bigg]$$

$$= -\frac{\pi\sqrt{3}}{40960} + \frac{13\pi}{221184} - \frac{1}{40960}\ln 3$$

$$+ \frac{1}{184320\sqrt{3}}\ln(2+\sqrt{3}) - \frac{1}{46080}.$$

（六）$\sum\limits_{n=1}^{\infty}\frac{1}{(12n-11)(12n-8)(12n-5)}$

$$= \frac{\pi}{108\sqrt{3}} + \frac{1}{36}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-5)(12n-2)}$$

$$= -\frac{\pi}{54\sqrt{3}} + \frac{\pi}{108} - \frac{1}{54}\ln 2 + \frac{1}{36\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-2)(12n+1)}$$

$$= \frac{\pi}{36\sqrt{3}} + \frac{1}{108}\ln 2 - \frac{1}{18}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-2)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\Bigg[\frac{1}{(12n-11)(12n-8)(12n-5)}$$

$$+ \frac{1}{(12n-5)(12n-2)(12n+1)}\Bigg]$$

$$= \frac{\pi}{27\sqrt{3}} + \frac{1}{27}\ln 2 - \frac{1}{18},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-2)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-8)(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-2)(12n+1)}\right]$$

$$=-\frac{\pi}{54\sqrt{3}}+\frac{1}{54}\ln 2+\frac{1}{18},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-8)(12n-5)(12n-2)}$$

$$=\frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-8)(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-8)(12n-5)(12n-2)}\right]$$

$$=\frac{\pi}{324\sqrt{3}}-\frac{\pi}{972}+\frac{5}{972}\ln 2$$

$$-\frac{1}{324\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-8)(12n-5)(12n-2)(12n+1)}$$

$$=-\frac{5\pi}{972\sqrt{3}}+\frac{\pi}{972}-\frac{1}{324}\ln 2$$

$$+\frac{1}{324\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{162};$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-8)(12n-5)(12n-2)(12n+1)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-8)(12n-5)(12n-2)}\right.$$

$$\left.-\frac{1}{(12n-8)(12n-5)(12n-2)(12n+1)}\right]$$

$$=\frac{\pi}{1458\sqrt{3}}-\frac{\pi}{5832}+\frac{1}{1458}\ln 2$$

$$-\frac{1}{1944\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{1944}.$$

例3 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(8n-7)(8n-6)(8n-5)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

解 于定理9令$a=1$，$b=2$，$k=16$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(16n-15)(16n-14)}=\sum_{n=1}^{\infty}\left(\frac{1}{16n-15}-\frac{1}{16n-14}\right)$$

$$=C(1,16)-\frac{1}{2}C(1,8)+\frac{1}{16}\ln 2.$$

根据定理6与8，有

$$C(1,16)-\frac{1}{2}C(1,8)=\frac{1}{2}A(1,8).$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(16n-15)(16n-14)}=\frac{1}{2}A(1,8)+\frac{1}{16}\ln 2.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(16n-14)(16n-13)}$$

$$=\frac{\pi}{16}-\frac{1}{16}\ln 2+\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{2}A(3,8),$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-13)(16n-12)}$$

$$=\frac{\pi}{16\sqrt{2}}-\frac{\pi}{16}+\frac{1}{8}\ln 2-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{2}A(3,8),$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-12)(16n-11)}$$

$$=\frac{\pi}{16\sqrt{2}}-\frac{1}{8}\ln 2+\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{2}A(5,8)$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-7)(16n-6)}$$

$$=\frac{\pi}{8\sqrt{2}}+\frac{1}{16}\ln 2+\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{2}A(1,8),$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-6)(16n-5)}$$

$$=-\frac{\pi}{8\sqrt{2}}+\frac{\pi}{16}-\frac{1}{16}\ln 2+\frac{1}{2}A(3,8),$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-5)(16n-4)}$$

$$=\frac{\pi}{16\sqrt{2}}+\frac{1}{8}\ln 2-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{2}A(3,8),$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-4)(16n-3)}$$

$$=\frac{\pi}{16\sqrt{2}}-\frac{\pi}{16}-\frac{1}{8}\ln 2+\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{2}A(5,8).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(16n-15)(16n-14)(16n-13)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(16n-15)(16n-14)}-\frac{1}{(16n-14)(16n-13)}\right]$$

$$=-\frac{\pi}{32}+\frac{1}{16}\ln 2-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{4}A(1,8)$$

$$+\frac{1}{4}A(3,8)=0.1668812545\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-14)(16n-13)(16n-12)}$$

$$=-\frac{\pi}{32\sqrt{2}}+\frac{\pi}{16}-\frac{3}{32}\ln 2+\frac{3}{16\sqrt{2}}\ln(\sqrt{2}+1)$$

$$-\frac{1}{2}A(3,8)=0.0418525697\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-13)(16n-12)(16n-11)}$$

$$=-\frac{\pi}{32}+\frac{1}{8}\ln 2-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{4}A(3,8)$$

$$+\frac{1}{4}A(5,8)=0.0168290778\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-7)(16n-6)(16n-5)}$$

$$=\frac{\pi}{8\sqrt{2}}-\frac{\pi}{32}+\frac{1}{16}\ln 2+\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)$$

$$-\frac{1}{4}A(1,8)-\frac{1}{4}A(3,8)=0.0010927858\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-6)(16n-5)(16n-4)}$$

$$=-\frac{3\pi}{32\sqrt{2}}+\frac{\pi}{32}-\frac{3}{32}\ln 2+\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{2}A(3,8)=0.00083277164\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-5)(16n-4)(16n-3)}$$

$$=\frac{\pi}{32}+\frac{1}{8}\ln 2-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{4}A(3,8)$$

$$-\frac{1}{4}A(5,8)=0.00065140722\cdots.$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-15)(16n-14)(16n-13)(16n-12)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\Big[\frac{1}{(16n-15)(16n-14)(16n-13)}$$

$$-\frac{1}{(16n-14)(16n-13)(16n-12)}\Big]$$

$$=\frac{\pi}{96\sqrt{2}}-\frac{\pi}{32}+\frac{5}{96}\ln 2-\frac{5}{48\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{12}A(1,8)+\frac{1}{4}A(3,8)=0.0416762282\cdots;$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-14)(16n-13)(16n-12)(16n-11)}$$

$$=-\frac{\pi}{96\sqrt{2}}+\frac{\pi}{32}-\frac{7}{96}\ln 2+\frac{5}{48\sqrt{2}}\ln(\sqrt{2}+1)$$

$$-\frac{1}{4}A(3,8)-\frac{1}{12}A(5,8)=0.0083411639\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)}$$

$$=\frac{7\pi}{96\sqrt{2}}-\frac{\pi}{48}+\frac{5}{96}\ln 2+\frac{1}{48\sqrt{2}}\ln(\sqrt{2}+1)$$

$$-\frac{1}{12}A(1,8)-\frac{1}{4}A(3,8)=0.000086671386\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-6)(16n-5)(16n-4)(16n-3)}$$

$$=-\frac{\pi}{32\sqrt{2}}-\frac{7}{96}\ln 2+\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{4}A(3,8)+\frac{1}{12}A(5,8)=0.000060454803\cdots.$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-15)(16n-14)(16n-13)\cdots(16n-11)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{(16n-15)(16n-14)(16n-13)(16n-12)}\right.$$

$$\left.-\frac{1}{(16n-14)(16n-13)(16n-12)(16n-11)}\right]$$

$$=\frac{\pi}{192\sqrt{2}}-\frac{\pi}{64}+\frac{1}{32}\ln 2-\frac{5}{96\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{48}A(1,8)+\frac{1}{8}A(3,8)+\frac{1}{48}A(5,8)$$

$$=0.0083337660\cdots,$$

$$\sum_{n=1}^{\infty}\frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)(16n-3)}$$

$$=\frac{5\pi}{192\sqrt{2}}-\frac{\pi}{192}+\frac{1}{32}\ln 2-\frac{1}{96\sqrt{2}}\ln(\sqrt{2}+1)$$

$$-\frac{1}{48}A(1,8)-\frac{1}{8}A(3,8)-\frac{1}{48}A(5,8)$$

$$=0.0000065541448\cdots.$$

于是 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(8n-7)(8n-6)(8n-5)}$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(16n-15)(16n-14)(15n-13)} - \frac{1}{(16n-7)(16n-6)(16n-5)}\right]$$

$$= -\frac{\pi}{8\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1) + \frac{1}{2}A(1,8) + \frac{1}{2}A(3,8) = 0.1657884687\cdots,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(16n-15)(16n-14)(16n-13)(16n-12)} - \frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)}\right]$$

$$= -\frac{\pi}{16\sqrt{2}} - \frac{\pi}{96} - \frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1) + \frac{1}{6}A(1,8) + \frac{1}{2}A(3,8) = 0.0415895569\cdots,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(16n-15)(16n-14)(16n-13)\cdots(16n-11)} - \frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)(16n-3)}\right]$$

$$= -\frac{\pi}{48\sqrt{2}} - \frac{\pi}{96} - \frac{1}{24\sqrt{2}}\ln(\sqrt{2}+1) + \frac{1}{24}A(1,8) + \frac{1}{4}A(3,8) + \frac{1}{24}A(5,8) = 0.0083272119\cdots.$$

例4 求和：

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-7}-\frac{1}{12n-5}-\frac{1}{12n-1}\right),$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{1}{12n-7}-\frac{1}{12n-5}+\frac{1}{12n-1}\right).$$

解 于定理9令 $a=1$，$b=7$，$k=12$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{1}{12n-5}\right)=C(1,12)-C(7,12)+\frac{1}{12}\ln 7.$$

根据定理6，有

$$C(1,12)-C(7,12)=\frac{\pi}{6}-\frac{1}{12}\ln 7+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{1}{12n-5}\right)=\frac{\pi}{6}+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-7}-\frac{1}{12n-1}\right)=\frac{\pi}{6}-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-7}-\frac{1}{12n-5}-\frac{1}{12n-1}\right)=\frac{\pi}{3},$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{1}{12n-7}-\frac{1}{12n-5}+\frac{1}{12n-1}\right)$$

$$=\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}\right)=\frac{\pi}{4}+\frac{1}{2}\ln 2,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-2}-\frac{1}{4n-1}+\frac{1}{4n}\right)=\frac{\pi}{4}-\frac{1}{2}\ln 2;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}+\frac{1}{8n-5}-\frac{1}{8n-3}-\frac{1}{8n-1}\right)=\frac{\pi}{2\sqrt{2}},$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-5}-\frac{1}{8n-3}+\frac{1}{8n-1}\right)$$

$$=\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}+\frac{1}{6n-4}+\frac{1}{6n-3}-\frac{1}{6n-2}-\frac{1}{6n-1}-\frac{1}{6n}\right)$$

$$=\frac{2\pi}{3\sqrt{3}}+\frac{1}{3}\ln 2;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-9}+\frac{1}{12n-7}-\frac{1}{12n-5}-\frac{1}{12n-3}\right.$$

$$\left.-\frac{1}{12n-1}\right)=\frac{5\pi}{12};$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}+\frac{1}{8n-6}+\frac{1}{8n-5}+\frac{1}{8n-4}-\frac{1}{8n-3}\right.$$

$$\left.-\frac{1}{8n-2}-\frac{1}{8n-1}-\frac{1}{8n}\right)$$

$$=\frac{\pi}{2\sqrt{2}}+\frac{\pi}{8}+\frac{1}{4}\ln 2,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}+\frac{1}{8n-6}-\frac{1}{8n-5}-\frac{1}{8n-4}-\frac{1}{8n-3}\right.$$

$$\left.-\frac{1}{8n-2}+\frac{1}{8n-1}+\frac{1}{8n}\right)$$

$$=\frac{\pi}{8}-\frac{1}{4}\ln 2+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-10}+\cdots+\frac{1}{12n-7}-\frac{1}{12n-6}\right.$$
$$\left.-\frac{1}{12n-5}-\cdots-\frac{1}{12n-1}\right)$$
$$=\frac{5\pi}{12}+\frac{\pi}{3\sqrt{3}},$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-10}+\cdots+\frac{1}{12n-6}-\frac{1}{12n-5}\right.$$
$$\left.-\frac{1}{12n-4}-\cdots-\frac{1}{12n}\right)$$
$$=\frac{5\pi}{12}+\frac{\pi}{3\sqrt{3}}+\frac{1}{6}\ln 2,$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-10}+\frac{1}{12n-9}-\frac{1}{12n-8}-\frac{1}{12n-7}\right.$$
$$\left.-\cdots-\frac{1}{12n-3}+\frac{1}{12n-2}+\frac{1}{12n-1}+\frac{1}{12n}\right)$$
$$=\frac{\pi}{12}+\frac{1}{6}\ln 2+\frac{1}{\sqrt{3}}\ln(2+\sqrt{3});$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{16n-15}-\frac{1}{16n-13}+\frac{1}{16n-7}-\frac{1}{16n-5}\right)$$
$$=\frac{\pi}{8}+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{16n-11}-\frac{1}{16n-9}+\frac{1}{16n-3}-\frac{1}{16n-1}\right)$$
$$=\frac{\pi}{8}-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{16n-15}-\frac{1}{16n-9}+\frac{1}{16n-7}-\frac{1}{16n-1}\right)$$
$$=\frac{\pi}{4\sqrt{2}}+\frac{\pi}{8},$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{16n-13}-\frac{1}{16n-11}+\frac{1}{16n-5}-\frac{1}{16n-3}\right)$$

$$=\frac{\pi}{4\sqrt{2}}-\frac{\pi}{8}.$$

例5 求和：

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{1}{24n-19}+\frac{1}{24n-11}-\frac{1}{24n-7}\right).$$

解 于定理9令 $a=1$，$b=5$，$k=24$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{1}{24n-19}\right)=C(1,24)-C(5,24)+\frac{1}{24}\ln 5$$

$$=\left[C(1,24)-\frac{1}{2}C(1,12)\right]-\left[C(5,24)-\frac{1}{2}C(5,12)\right]+\frac{1}{2}C(1,12)-\frac{1}{2}C(5,12)+\frac{1}{24}\ln 5.$$

根据定理5，有

$$C(1,24)-\frac{1}{2}C(1,12)=\frac{1}{2}A(1,12),$$

$$C(5,24)-\frac{1}{2}C(5,12)=\frac{1}{2}A(5,12).$$

于是 $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{1}{24n-19}\right)$

$$=\frac{1}{2}A(1,12)-\frac{1}{2}A(5,12)+\frac{1}{2}C(1,12)-\frac{1}{2}C(5,12)+\frac{1}{24}\ln 5.$$

于定理9令 $a=13$，$b=17$，$k=24$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-11}-\frac{1}{24n-7}\right)$$

$$=C(13,24)-C(17,24)+\frac{1}{24}\ln\frac{17}{13}$$

$$=\left[C(13,24)-\frac{1}{2}C(13,12)\right]$$

$$-\left[C(17,24)-\frac{1}{2}C(17,12)\right]+\frac{1}{2}C(13,12)$$

$$-\frac{1}{2}C(17,12)+\frac{1}{24}\ln\frac{17}{13}$$

$$=\frac{1}{2}A(13,12)-\frac{1}{2}A(17,12)+\frac{1}{2}C(13,12)$$

$$-\frac{1}{2}C(17,12)+\frac{1}{24}\ln\frac{17}{13}.$$

$$A(13,12)=-A(1,12)+1,$$

$$A(17,12)=-A(5,12)+\frac{1}{5},$$

$$\frac{1}{2}A(13,12)-\frac{1}{2}A(17,12)$$

$$=-\frac{1}{2}A(1,12)+\frac{1}{2}A(5,12)+\frac{2}{5},$$

于是 $$\sum_{n=1}^{\infty}\left(\frac{1}{24n-11}-\frac{1}{24n-7}\right)$$

$$=-\frac{1}{2}A(1,12)+\frac{1}{2}A(5,12)+\frac{2}{5}+\frac{1}{2}C(13,12)$$

$$-\frac{1}{2}C(17,12)+\frac{1}{24}\ln\frac{17}{13}.$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{1}{24n-19}+\frac{1}{24n-11}-\frac{1}{24n-7}\right)$$

$$=\frac{1}{2}C(1,12)-\frac{1}{2}C(5,12)+\frac{1}{2}C(13,12)$$

$$-\frac{1}{2}C(17,12)+\frac{1}{24}\ln\frac{5\cdot 17}{13}+\frac{2}{5}.$$

于命题2令　$a=1$, $b=1$, $k=12$, 则有

$$C(13,12)=C(1,12)+\frac{1}{12}\ln 13-1,$$

令　$a=5$, $b=1$, $k=12$, 则有

$$C(17,12)=C(5,12)+\frac{1}{12}\ln\frac{17}{5}-\frac{1}{5}.$$

于是　$\frac{1}{2}C(13,12)-\frac{1}{2}C(17,12)$

$$=\frac{1}{2}C(1,12)-\frac{1}{2}C(5,12)+\frac{1}{24}\ln\frac{5\cdot 13}{17}-\frac{2}{5}.$$

从而　$\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{1}{24n-19}+\frac{1}{24n-11}-\frac{1}{24n-7}\right)$

$$=C(1,12)-C(5,12)+\frac{1}{12}\ln 5.$$

根据定理6，有

$$C(1,12)-C(5,12)=\frac{\pi}{4\sqrt{3}}-\frac{1}{12}\ln 5$$

$$+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}).$$

所以　$\sum_{n=1}^{\infty}\left(\frac{1}{24n-28}-\frac{1}{24n-19}+\frac{1}{24n-11}-\frac{1}{24n-7}\right)$

$$=\frac{\pi}{4\sqrt{3}}+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-17}-\frac{1}{24n-13}+\frac{1}{24n-5}-\frac{1}{24n-1}\right)$$

$$=\frac{\pi}{4\sqrt{3}}-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{1}{24n-13}+\frac{1}{24n-11}-\frac{1}{24n-1}\right)$$

$$=\frac{\pi}{4\sqrt{3}}+\frac{\pi}{6},$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-19}-\frac{1}{24n-17}+\frac{1}{24n-7}-\frac{1}{24n-5}\right)$$

$$=-\frac{\pi}{4\sqrt{3}}+\frac{\pi}{6}.$$

2. $\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{B-A}{k(n-1)+d}\right]$

定理10 设 A、B、b、d 都是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{B-A}{k(n-1)+d}\right]$$

$$=AC(a,k)-BC(b,k)+(B-A)C(d,k)$$

$$+\frac{1}{k}\ln\frac{b^{B}}{a^{A}d^{B-A}}.$$

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{A}{k(i-1)+a}-\frac{B}{k(i-1)+b}+\frac{B-A}{k(i-1)+d}\right]$$

$$=AC(a,k)-BC(b,k)+(B-A)C(d,k)$$

$$+\frac{1}{k}\ln\frac{(nk-k+a)^{A}(nk-k+d)^{B-A}}{(nk-k+b)^{B}}$$

$$+\frac{1}{k}\ln\frac{b^B}{a^A d^{B-A}}+A\varepsilon_n(a,k)-B\varepsilon_n(b,k)$$

$$+(B-A)\varepsilon_n(d,k).$$

令 $n\to\infty$ 并取极限，定理即得证。

例1 求和 $\sum\limits_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)(8n-1)}$.

解 用分项分式方法求得

$$\frac{1}{(8n-7)(8n-4)(8n-1)}$$

$$=\frac{1}{18}\left(\frac{1}{8n-7}-\frac{2}{8n-4}+\frac{1}{8n-1}\right).$$

于定理10令 $A=1$，$B=2$，$a=1$，$b=4$，$d=7$，$k=8$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)(8n-1)}$$

$$=\frac{1}{18}\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{2}{8n-4}+\frac{1}{8n-1}\right)$$

$$=\frac{1}{18}C(1,8)-\frac{1}{36}C(1,2)+\frac{1}{18}C(7,8)+\frac{1}{144}\ln\frac{2^4}{7}.$$

根据定理 6，有

$$\frac{1}{18}C(1,8)-\frac{1}{72}C(1,2)$$

$$=\frac{\pi}{144\sqrt{2}}+\frac{\pi}{288}+\frac{1}{72\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{1}{18}C(7,8)-\frac{1}{72}C(1,2)$$

$$=-\frac{\pi}{144\sqrt{2}}-\frac{\pi}{288}+\frac{1}{144}\ln 7$$

$$+\frac{1}{72\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)(8n-1)}$$

$$=\frac{1}{36}\ln 2+\frac{1}{36\sqrt{2}}\ln(\sqrt{2}+1).$$

例2 求和 $\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)\cdots(6n-1)}$

解 用分项分式方法求得

$$\frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$=\frac{1}{12}\left(\frac{5}{6n-5}-\frac{8}{6n-4}+\frac{3}{6n-3}\right).$$

于定理10令 $A=5$, $B=8$, $a=1$, $b=2$, $d=3$, $k=6$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{5}{6n-5}-\frac{8}{6n-4}+\frac{3}{6n-3}\right)$$

$$=\frac{5}{12}C(1,6)-\frac{1}{3}C(1,3)+\frac{1}{12}C(1,2)+\frac{1}{72}\ln\frac{2^8}{3^3}.$$

根据定理6，有

$$\frac{5}{12}C(1,6)-\frac{5}{24}C(1,3)=\frac{5\pi}{72\sqrt{3}}+\frac{5}{72}\ln 2,$$

$$\frac{1}{12}C(1,2)-\frac{1}{8}C(1,3)=-\frac{\pi}{48\sqrt{3}}+\frac{1}{48}\ln\frac{2^2}{3}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$=\frac{7\pi}{144\sqrt{3}}+\frac{2}{9}\ln 2-\frac{1}{16}\ln 3.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(6n-3)(6n-2)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{2}{6n-4}-\frac{3}{6n-3}+\frac{1}{6n-2}\right)$$

$$=\frac{\pi}{72\sqrt{3}}+\frac{1}{8}\ln 3-\frac{1}{6}\ln 2;$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-3)(6n-2)(6n-1)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{3}{6n-3}-\frac{4}{6n-2}+\frac{1}{6n-1}\right)$$

$$=\frac{\pi}{144\sqrt{3}}+\frac{1}{9}\ln 2-\frac{1}{16}\ln 3.$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(6n-5)(6n-4)(6n-3)}\right.$$

$$\left.-\frac{n}{(6n-4)(6n-3)(6n-2)}\right]$$

$$=\frac{5\pi}{432\sqrt{3}}+\frac{7}{54}\ln 2-\frac{1}{16}\ln 3;$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(6n-4)(6n-3)(6n-2)}\right.$$

$$-\frac{n}{(6n-3)(6n-2)(6n-1)}\Bigg]$$

$$=\frac{\pi}{432\sqrt{3}}+\frac{1}{16}\ln 3-\frac{5}{54}\ln 2;$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\Bigg[\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$-\frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)}\Bigg]$$

$$=\frac{\pi}{432\sqrt{3}}+\frac{1}{18}\ln 2-\frac{1}{32}\ln 3.$$

仿此求得

(一) $$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{7}{8n-7}-\frac{12}{8n-6}+\frac{5}{8n-5}\right)$$

$$=\frac{3\pi}{32\sqrt{2}}-\frac{5\pi}{128}+\frac{3}{32}\ln 2+\frac{1}{32\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-6)(8n-5)(8n-4)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{6}{8n-6}-\frac{10}{8n-5}+\frac{4}{8n-4}\right)$$

$$=-\frac{5\pi}{64\sqrt{2}}+\frac{\pi}{16}-\frac{7}{64}\ln 2+\frac{5}{32\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{5}{8n-5}-\frac{8}{8n-4}+\frac{3}{8n-3}\right)$$

$$=\frac{\pi}{64\sqrt{2}}-\frac{\pi}{128}+\frac{1}{8}\ln 2-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-4)(8n-3)(8n-2)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{4}{8n-4}-\frac{6}{8n-3}+\frac{2}{8n-2}\right)$$

$$=\frac{3\pi}{64\sqrt{2}}-\frac{\pi}{32}-\frac{5}{64}\ln 2+\frac{3}{32\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{3}{8n-3}-\frac{4}{8n-2}+\frac{1}{8n-1}\right)$$

$$=-\frac{\pi}{32\sqrt{2}}+\frac{3\pi}{128}+\frac{1}{32}\ln 2-\frac{1}{32\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(4n-3)(4n-2)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-7)(8n-6)(8n-5)}-\frac{2n}{(8n-3)(8n-2)(8n-1)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)}-\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)}-2\sum_{n=1}^{\infty}\frac{n}{(8n-3)(8n-2)(8n-1)}$$

$$=\frac{\pi}{8\sqrt{2}}-\frac{\pi}{16}+\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-2)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-7)(8n-6)(8n-5)}\right.$$

$$\left.+\frac{2n}{(8n-3)(8n-2)(8n-1)}\right]$$

$$=\frac{\pi}{32}+\frac{1}{8}\ln 2;$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(8n-7)(8n-6)(8n-5)}\right.$$

$$\left.-\frac{n}{(8n-6)(8n-5)(8n-4)}\right]$$

$$=\frac{11\pi}{192\sqrt{2}}-\frac{13\pi}{384}+\frac{13}{192}\ln 2-\frac{1}{24\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=-\frac{\pi}{32\sqrt{2}}+\frac{3\pi}{128}-\frac{5}{64}\ln 2+\frac{3}{32\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=-\frac{\pi}{96\sqrt{2}}+\frac{\pi}{128}+\frac{13}{192}\ln 2-\frac{7}{96\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{5\pi}{192\sqrt{2}}-\frac{7\pi}{384}-\frac{7}{192}\ln 2+\frac{1}{24\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)}\right.$$

$$\left.-\frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)}\right]$$

$$=\frac{17\pi}{768\sqrt{2}}-\frac{11\pi}{768}+\frac{7}{192}\ln 2$$

$$-\frac{13}{384\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=-\frac{\pi}{192\sqrt{2}}+\frac{\pi}{256}-\frac{7}{192}\ln 2$$

$$+\frac{1}{24\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=-\frac{7\pi}{768\sqrt{2}}+\frac{5\pi}{768}+\frac{5}{192}\ln 2$$

$$-\frac{11}{384\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}\right.$$

$$\left.-\frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}\right]$$

$$=\frac{7\pi}{1280\sqrt{2}}-\frac{7\pi}{1920}+\frac{7}{480}\ln 2-\frac{29}{1920\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$=\frac{\pi}{1280\sqrt{2}}-\frac{\pi}{1920}-\frac{1}{80}\ln 2+\frac{9}{640\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}\right.$$

$$\left.-\frac{n}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}\right]$$

$$=\frac{\pi}{1280\sqrt{2}}-\frac{\pi}{1920}+\frac{13}{2880}\ln 2$$

$$-\frac{7}{1440\sqrt{2}}\ln(\sqrt{2}+1).$$

(二) $$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-3)}$$

$$=\frac{1}{64}\sum_{n=1}^{\infty}\left(\frac{7}{8n-7}-\frac{10}{8n-5}+\frac{3}{8n-3}\right)$$

$$=-\frac{3\pi}{256\sqrt{2}}+\frac{5\pi}{256}+\frac{7}{128\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-5)(8n-3)(8n-1)}$$

$$=\frac{1}{64}\sum_{n=1}^{\infty}\left(\frac{5}{8n-5}-\frac{6}{8n-3}+\frac{1}{8n-1}\right)$$

$$=\frac{5\pi}{256\sqrt{2}}-\frac{3\pi}{256}+\frac{1}{128\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{64}\sum_{n=1}^{\infty}\left(\frac{3}{8n-3}-\frac{2}{8n-1}-\frac{1}{8n+1}\right)$$

$$= -\frac{\pi}{256\sqrt{2}} + \frac{\pi}{256} - \frac{3}{128\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{64}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(4n-3)(4n-1)(4n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-7)(8n-5)(8n-3)}\right.$$

$$\left. - \frac{2n}{(8n-3)(8n-1)(8n+1)}\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-3)}$$

$$- \sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(8n-3)}$$

$$- 2\sum_{n=1}^{\infty}\frac{n}{(8n-3)(8n-1)(8n+1)}$$

$$= \frac{\pi}{64\sqrt{2}} + \frac{3}{32\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{32},$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(4n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-7)(8n-5)(8n-3)}\right.$$

$$\left. + \frac{2n}{(8n-5)(8n-3)(8n-1)}\right]$$

$$= \frac{\pi}{64} + \frac{1}{32};$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-3)(8n-1)}$$

$$= \frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n}{(8n-7)(8n-5)(8n-3)}\right.$$

$$-\frac{n}{(8n-5)(8n-3)(8n-1)}\Big]$$

$$=-\frac{\pi}{192\sqrt{2}}+\frac{\pi}{192}+\frac{1}{128\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{\pi}{256\sqrt{2}}-\frac{\pi}{384}+\frac{1}{192\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{384},$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\Big[\frac{n}{(8n-7)(8n-5)(8n-3)(8n-1)}$$

$$-\frac{n}{(8n-5)(8n-3)(8n-1)(8n+1)}\Big]$$

$$=-\frac{7\pi}{6144\sqrt{2}}+\frac{\pi}{1024}+\frac{1}{3072\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{3072}.$$

（三）$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{20}{12n-10}+\frac{9}{12n-9}\right)$$

$$=-\frac{\pi\sqrt{3}}{64}+\frac{31\pi}{576}+\frac{5}{72}\ln 2-\frac{3}{64}\ln 3$$

$$+\frac{11}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{5}{12n-10}-\frac{9}{12n-9}+\frac{4}{12n-8}\right)$$

$$=\frac{19\pi}{288\sqrt{3}}-\frac{\pi}{32}+\frac{3}{32}\ln 3-\frac{17}{144}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{9}{12n-9}-\frac{16}{12n-8}+\frac{7}{12n-7}\right)$$

$$=-\frac{37\pi}{576\sqrt{3}}+\frac{23\pi}{576}+\frac{1}{6}\ln 2-\frac{3}{64}\ln 3$$

$$-\frac{7}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-8)(12n-7)(12n-6)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{4}{12n-8}-\frac{7}{12n-7}+\frac{3}{12n-6}\right)$$

$$=\frac{25\pi}{288\sqrt{3}}-\frac{7\pi}{144}-\frac{5}{48}\ln 2-\frac{1}{32}\ln 3$$

$$+\frac{7}{48\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-6)(12n-5)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{7}{12n-7}-\frac{12}{12n-6}+\frac{5}{12n-5}\right)$$

$$=-\frac{\pi}{96\sqrt{3}}+\frac{\pi}{144}+\frac{1}{16}\ln 3+\frac{1}{24}\ln 2$$

$$-\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-6)(12n-5)(12n-4)}$$

$$=\frac{1}{12}\sum\left(\frac{3}{12n-6}-\frac{5}{12n-5}+\frac{2}{12n-4}\right)$$

$$= -\frac{17\pi}{288\sqrt{3}} + \frac{5\pi}{144} - \frac{1}{16}\ln 2 - \frac{1}{32}\ln 3$$

$$+\frac{5}{48\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{8}{12n-4}+\frac{3}{12n-3}\right)$$

$$=\frac{23\pi}{576\sqrt{3}}-\frac{13\pi}{576}+\frac{1}{12}\ln 2-\frac{1}{64}\ln 3$$

$$-\frac{5}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-4)(12n-3)(12n-2)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{2}{12n-4}-\frac{3}{12n-3}+\frac{1}{12n-2}\right)$$

$$=-\frac{5\pi}{288\sqrt{3}}+\frac{\pi}{96}+\frac{1}{32}\ln 3-\frac{7}{144}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-3)(12n-2)(12n-1)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{3}{12n-3}-\frac{4}{12n-2}+\frac{1}{12n-1}\right)$$

$$=\frac{\pi}{64\sqrt{3}}-\frac{5\pi}{576}+\frac{1}{72}\ln 2-\frac{1}{64}\ln 3$$

$$+\frac{1}{96\sqrt{3}}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)}\right.$$

$$\left. - \frac{2n}{(12n-5)(12n-4)(12n-3)}\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)}$$

$$- \sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)}$$

$$- 2\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)}$$

$$= -\frac{\pi}{9\sqrt{3}} + \frac{13\pi}{144} - \frac{1}{9}\ln 2 + \frac{5}{24\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$= \frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)}\right.$$

$$\left. - \frac{n}{(12n-10)(12n-9)(12n-8)}\right]$$

$$= -\frac{65\pi}{1728\sqrt{3}} + \frac{49\pi}{1728} + \frac{1}{16}\ln 2 - \frac{3}{64}\ln 3$$

$$+ \frac{11}{288\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{25\pi}{576\sqrt{3}} - \frac{41\pi}{1728} + \frac{3}{64}\ln 3 - \frac{41}{432}\ln 2$$

$$+ \frac{7}{288\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= -\frac{29\pi}{576\sqrt{3}} + \frac{17\pi}{576} + \frac{13}{144}\ln 2 - \frac{1}{192}\ln 3$$

$$-\frac{7}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= \frac{7\pi}{216\sqrt{3}} - \frac{\pi}{54} - \frac{7}{144}\ln 2 - \frac{1}{32}\ln 3$$

$$+\frac{13}{144\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= \frac{7\pi}{432\sqrt{3}} - \frac{\pi}{108} + \frac{1}{32}\ln 3 + \frac{5}{144}\ln 2$$

$$-\frac{11}{144\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$= -\frac{19\pi}{576\sqrt{3}} + \frac{11\pi}{576} - \frac{7}{144}\ln 2 - \frac{1}{192}\ln 3$$

$$+\frac{5}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= \frac{11\pi}{576\sqrt{3}} - \frac{19\pi}{1728} + \frac{19}{432}\ln 2 - \frac{1}{64}\ln 3$$

$$-\frac{5}{288\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$= -\frac{19\pi}{1728\sqrt{3}} + \frac{11\pi}{1728} + \frac{1}{64}\ln 3 - \frac{1}{48}\ln 2$$

$$-\frac{1}{288\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)}\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= -\frac{7\pi}{108\sqrt{3}} + \frac{19\pi}{432} - \frac{5}{108}\ln 2$$

$$+\frac{5}{72\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)}\right.$$

$$\left.-\frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)}\right]$$

$$= -\frac{35\pi}{1728\sqrt{3}} + \frac{5\pi}{384} + \frac{17}{432}\ln 2 - \frac{3}{128}\ln 3$$

$$+\frac{1}{288\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$=\frac{\pi\sqrt{3}}{128}-\frac{23\pi}{1728}+\frac{5}{384}\ln 3-\frac{5}{108}\ln 2$$

$$+\frac{7}{288\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$=-\frac{143\pi}{6912\sqrt{3}}+\frac{83\pi}{6912}+\frac{5}{144}\ln 2+\frac{5}{768}\ln 3$$

$$-\frac{47}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-8)(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$=\frac{7\pi}{1728\sqrt{3}}-\frac{\pi}{432}-\frac{1}{48}\ln 2-\frac{1}{64}\ln 3$$

$$+\frac{1}{24\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$=\frac{85\pi}{6912\sqrt{3}}-\frac{49\pi}{6912}+\frac{7}{768}\ln 3+\frac{1}{48}\ln 2$$

$$-\frac{37}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-6)(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$=-\frac{5\pi}{384\sqrt{3}}+\frac{13\pi}{1728}-\frac{5}{216}\ln 2+\frac{1}{384}\ln 3$$

$$+\frac{5}{288\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$=\frac{13\pi}{1728\sqrt{3}}-\frac{5\pi}{1152}+\frac{7}{432}\ln 2-\frac{1}{128}\ln 3$$

$$-\frac{1}{288\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)\cdots(12n-1)}$$

$$=-\frac{\pi}{36\sqrt{3}}+\frac{5\pi}{288}-\frac{1}{108}\ln 2+\frac{1}{72\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}\right.$$

$$-\frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}\Bigg]$$

$$=-\frac{151\pi}{17280\sqrt{3}}+\frac{91\pi}{17280}+\frac{37}{2160}\ln 2-\frac{7}{960}\ln 3$$

$$-\frac{1}{240\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}$$

$$=\frac{61\pi}{6912\sqrt{3}}-\frac{35\pi}{6912}+\frac{1}{768}\ln 3-\frac{7}{432}\ln 2$$

$$+\frac{5}{384\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-4)}$$

$$=-\frac{19\pi}{3840\sqrt{3}}+\frac{11\pi}{3840}+\frac{1}{90}\ln 2+\frac{17}{3840}\ln 3$$

$$-\frac{19}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-8)(12n-7)(12n-6)\cdots(12n-3)}$$

$$=-\frac{19\pi}{11520\sqrt{3}}+\frac{11\pi}{11520}-\frac{1}{120}\ln 2-\frac{19}{3840}\ln 3$$

$$+\frac{17}{1152\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-6)(12n-5)\cdots(12n-2)}$$

$$=\frac{35\pi}{6912\sqrt{3}}-\frac{101\pi}{34560}+\frac{1}{768}\ln 3+\frac{19}{2160}\ln 2$$

$$-\frac{19}{1920\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-6)(12n-5)(12n-4)\cdots(12n-1)}$$

$$=-\frac{71\pi}{17280\sqrt{3}}+\frac{41\pi}{17280}+\frac{1}{480}\ln 3-\frac{17}{2160}\ln 2$$

$$+\frac{1}{240\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}\right.$$

$$\left.-\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}\right]$$

$$=-\frac{607\pi}{207360\sqrt{3}}+\frac{119\pi}{69120}+\frac{1}{180}\ln 2$$

$$-\frac{11}{7680}\ln 3-\frac{11}{3840\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)}$$

$$=\frac{476\pi}{207360\sqrt{3}}-\frac{274\pi}{207360}-\frac{59}{12960}\ln 2-\frac{1}{1920}\ln 3$$

$$+\frac{17}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-3)}$$

$$=-\frac{19\pi}{34560\sqrt{3}}+\frac{11\pi}{34560}+\frac{7}{2160}\ln 2+\frac{1}{640}\ln 3$$

$$-\frac{1}{192\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-8)(12n-7)(12n-6)\cdots(12n-2)}$$

$$= -\frac{29\pi}{25920\sqrt{3}} + \frac{67\pi}{103680} - \frac{37}{12960}\ln 2 - \frac{1}{960}\ln 3$$

$$+ \frac{71}{17280\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-6)(12n-5)\cdots(12n-1)}$$

$$= \frac{317\pi}{207360\sqrt{3}} - \frac{61\pi}{69120} + \frac{1}{360}\ln 2 - \frac{1}{7680}\ln 3$$

$$-\frac{\sqrt{3}}{1280}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)}$$

$$= \frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}\right.$$

$$\left. - \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)}\right]$$

$$= -\frac{361\pi}{483840\sqrt{3}} + \frac{631\pi}{1451520} + \frac{131}{90720}\ln 2 - \frac{1}{7680}\ln 3$$

$$-\frac{269}{241920\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)}$$

$$= \frac{59\pi}{145152\sqrt{3}} - \frac{17\pi}{72576} - \frac{101}{90720}\ln 2 - \frac{1}{3360}\ln 3$$

$$+ \frac{5}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-2)}$$

$$= \frac{59\pi}{725760\sqrt{3}} - \frac{17\pi}{362880} + \frac{79}{90720}\ln 2 + \frac{1}{2688}\ln 3$$

$$-\frac{161}{120960\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-8)(12n-7)(12n-6)\cdots(12n-1)}$$

$$=-\frac{61\pi}{161280\sqrt{3}}+\frac{317\pi}{1451520}-\frac{73}{90720}\ln 2-\frac{1}{7680}\ln 3$$

$$+\frac{223}{241920\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\Big[\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)}$$

$$-\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)}\Big]$$

$$=-\frac{239\pi}{1658880\sqrt{3}}+\frac{971\pi}{11612160}+\frac{29}{90720}\ln 2$$

$$+\frac{3}{143360}\ln 3-\frac{619}{1935360\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}$$

$$=\frac{59\pi}{1451520\sqrt{3}}-\frac{17\pi}{725760}-\frac{1}{4032}\ln 2-\frac{3}{35840}\ln 3$$

$$+\frac{1}{2880\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-1)}$$

$$=\frac{667\pi}{11612160\sqrt{3}}-\frac{11\pi}{331776}+\frac{19}{90720}\ln 2+\frac{9}{143360}\ln 3$$

$$-\frac{109}{387072\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)}$$

$$=\frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}\right.$$

$$\left.-\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}\right]$$

$$=-\frac{143\pi}{6967296\sqrt{3}}+\frac{1243\pi}{104509440}+\frac{103}{1632960}\ln 2$$

$$+\frac{1}{86016}\ln 3-\frac{1291}{17418240\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)}$$

$$=-\frac{13\pi}{6967296\sqrt{3}}+\frac{113\pi}{104509440}-\frac{83}{1632960}\ln 2$$

$$-\frac{1}{61440}\ln 3+\frac{1217}{17418240\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-1)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)}\right.$$

$$\left.-\frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)}\right]$$

$$=-\frac{13\pi}{6967296\sqrt{3}}+\frac{113\pi}{104509440}+\frac{31}{2721600}\ln 2$$

$$+\frac{1}{358400}\ln 3-\frac{209}{14515200\sqrt{3}}\ln(2+\sqrt{3}).$$

（四）$\displaystyle\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)}$

$$=\frac{1}{96}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{18}{12n-9}+\frac{7}{12n-7}\right)$$

$$=\frac{\pi}{192\sqrt{3}}+\frac{\pi}{128}+\frac{3}{128}\ln 3+\frac{1}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-7)(12n-5)}$$

$$=\frac{1}{96}\sum_{n=1}^{\infty}\left(\frac{9}{12n-9}-\frac{14}{12n-7}+\frac{5}{12n-5}\right)$$

$$=\frac{19\pi}{768\sqrt{3}}-\frac{29\pi}{2304}-\frac{3}{256}\ln 3+\frac{\sqrt{3}}{128}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-5)(12n-3)}$$

$$=\frac{1}{96}\sum_{n=1}^{\infty}\left(\frac{7}{12n-7}-\frac{10}{12n-5}+\frac{3}{12n-3}\right)$$

$$=-\frac{17\pi}{768\sqrt{3}}+\frac{31\pi}{2304}-\frac{1}{256}\ln 3$$

$$+\frac{1}{128\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-3)(12n-1)}$$

$$=\frac{1}{96}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{6}{12n-3}+\frac{1}{12n-1}\right)$$

$$=\frac{\pi}{192\sqrt{3}}-\frac{\pi}{384}+\frac{1}{128}\ln 3-\frac{1}{96\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-3)(12n-1)(12n+1)}$$

$$=\frac{1}{96}\sum_{n=1}^{\infty}\left(\frac{3}{12n-3}-\frac{2}{12n-1}-\frac{1}{12n+1}\right)$$

$$= \frac{\pi}{768\sqrt{3}} - \frac{\pi}{2304} - \frac{1}{256}\ln 3 - \frac{1}{128\sqrt{3}}\ln(2+\sqrt{3})$$

$$+ \frac{1}{96}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-3)(6n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-9)(12n-7)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-3)(12n-1)}\right]$$

$$= 2\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-3)(12n-1)}$$

$$= \frac{\pi}{96} + \frac{1}{24\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$= \frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-9)(12n-7)}\right.$$

$$\left.-\frac{n}{(12n-9)(12n-7)(12n-5)}\right]$$

$$= -\frac{5\pi}{1536\sqrt{3}} + \frac{47\pi}{13824} + \frac{3}{512}\ln 3$$

$$-\frac{5}{2304\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$=\frac{\pi}{128\sqrt{3}}-\frac{5\pi}{1152}-\frac{1}{768}\ln 3$$

$$+\frac{1}{384\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$=-\frac{7\pi}{1536\sqrt{3}}+\frac{37\pi}{13824}-\frac{1}{512}\ln 3$$

$$+\frac{7}{2304\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{\pi}{1536\sqrt{3}}-\frac{5\pi}{13824}+\frac{1}{512}\ln 3$$

$$-\frac{1}{2304\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{576}.$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-3)(12n-1)(12n+1)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{5\pi}{3456}+\frac{1}{576\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{288},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)}\right]$$

$$=-\frac{17\pi}{12288\sqrt{3}}+\frac{107\pi}{110592}+\frac{11}{12288}\ln 3$$

$$-\frac{11}{18432\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$=\frac{19\pi}{12288\sqrt{3}}-\frac{97\pi}{110592}+\frac{1}{12288}\ln 3$$

$$-\frac{1}{18432\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-7)(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=-\frac{\pi}{1536\sqrt{3}}+\frac{7\pi}{18432}-\frac{1}{2048}\ln 3$$

$$+\frac{1}{2304\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{4608},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left[\frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n-3)}\right.$$

$$-\frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)}\Bigg]$$

$$=-\frac{\pi\sqrt{3}}{10240}+\frac{17\pi}{92160}+\frac{1}{12288}\ln 3$$

$$-\frac{1}{18432\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}$$

$$=\frac{3\pi\sqrt{3}}{40960}-\frac{139\pi}{1105920}+\frac{7}{122880}\ln 3$$

$$-\frac{1}{20480\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{46080};$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n+1)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\Bigg[\frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}$$

$$-\frac{n}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}\Bigg]$$

$$=-\frac{7\pi}{163840\sqrt{3}}+\frac{343\pi}{13271040}+\frac{1}{491520}\ln 3$$

$$-\frac{1}{2211840\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{552960}.$$

3. $$\sum_{n=1}^{\infty}\Bigg[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{D}{k(n-1)+d}$$

$$-\frac{A+D-B}{k(n-1)+e}\Bigg]$$

定理11 设A、B、D、b、d、e都是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{D}{k(n-1)+d}\right.$$
$$\left.-\frac{A+D-B}{k(n-1)+e}\right]$$
$$=AC(a,k)-BC(b,k)+DC(d,k)$$
$$-(A+D-B)C(e,k)+\frac{1}{k}\ln\frac{b^{B}e^{A+D-B}}{a^{A}d^{D}}.$$

证明 根据定理2，有

$$\sum_{i=1}^{\infty}\left[\frac{A}{k(i-1)+a}-\frac{B}{k(i-1)+b}+\frac{D}{k(i-1)+d}\right.$$
$$\left.-\frac{A+D-B}{k(i-1)+e}\right]$$
$$=AC(a,k)-BC(b,k)+DC(d,k)$$
$$-(A+D-B)C(e,k)$$
$$+\frac{1}{k}\ln\frac{(nk-k+a)^{A}(nk-k+d)^{D}}{(nk-k+b)^{B}(nk-k+e)^{A+D-B}}$$
$$+\frac{1}{k}\ln\frac{b^{B}e^{A+D-B}}{a^{A}d^{D}}+A\varepsilon_n(a,k)-B\varepsilon_n(b,k)$$
$$+D\varepsilon_n(d,k)-(A+D-B)\varepsilon_n(e,k),$$

令 $n\to\infty$ 并取极限，定理即得证

例1 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}.$$

解 用分项分式方法求得

$$\frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$$
$$=\frac{1}{288}\left(\frac{5}{6n-5}-\frac{9}{6n-3}+\frac{3}{6n-1}+\frac{1}{6n+1}\right).$$

于定理 11令 $A=5$, $B=9$, $D=3$, $a=1$, $b=3$, $d=5$, $e=7$, $k=6$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$=\frac{1}{288}\sum_{n=1}^{\infty}\left(\frac{5}{6n-5}-\frac{9}{6n-3}+\frac{3}{6n-1}+\frac{1}{6n+1}\right)$$

$$=\frac{5}{288}C(1,6)-\frac{1}{96}C(1,2)+\frac{1}{96}C(5,6)$$

$$+\frac{1}{288}C(7,6)+\frac{1}{1728}\ln\frac{3^9}{5^3 7}.$$

于命题 2 令 $a=1$, $b=1$, $k=6$, 则有

$$C(7,6)=C(1,6)+\frac{1}{6}\ln 7-1,$$

$$\frac{1}{288}C(7,6)=\frac{1}{288}C(1,6)+\frac{1}{1728}\ln 7-\frac{1}{288}.$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$

$$=\frac{1}{48}C(1,6)-\frac{1}{96}C(1,2)+\frac{1}{96}C(5,6)$$

$$+\frac{1}{1728}\ln\frac{3^9}{5^3}-\frac{1}{288}.$$

根据定理 6，有

$$\frac{1}{48}C(1,6)-\frac{1}{144}C(1,2)=\frac{\pi}{192\sqrt{3}}+\frac{1}{576}\ln 3,$$

$$\frac{1}{96}C(5,6)-\frac{1}{288}C(1,2)=-\frac{\pi}{384\sqrt{3}}+\frac{1}{1152}\ln 3\cdot 5^2.$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$

$$=\frac{\pi}{384\sqrt{3}}+\frac{1}{128}\ln 3-\frac{1}{288}.$$

例2 求和:

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}.$$

解 用分项分式方法求得

$$\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$=\frac{1}{216}\left(\frac{25}{6n-5}-\frac{48}{6n-4}+\frac{27}{6n-3}-\frac{4}{6n-2}\right).$$

于定理11令 $A=25$, $B=48$, $D=27$, $a=1$, $b=2$, $d=3$, $e=4$, $k=6$, 则有

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$=\frac{1}{216}\sum_{n=1}^{\infty}\left(\frac{25}{6n-5}-\frac{48}{6n-4}+\frac{72}{6n-3}-\frac{4}{6n-2}\right)$$

$$=\frac{25}{216}C(1,6)-\frac{1}{9}C(1,3)+\frac{1}{24}C(1,2)-\frac{1}{108}C(2,3)$$

$$+\frac{1}{1296}\ln\frac{2^{56}}{3^{27}}.$$

根据定理6，有

$$\frac{25}{216}C(1,6)-\frac{25}{432}C(1,3)=\frac{25\pi}{1296\sqrt{3}}+\frac{1}{1296}\ln 2^{25},$$

$$\frac{23}{648}C(1,2)-\frac{23}{432}C(1,3)=-\frac{23\pi}{2592\sqrt{3}}$$

$$+\frac{1}{2592}\ln\frac{2^{46}}{3^{23}},$$

$$\frac{1}{162}C(1,2)-\frac{1}{108}C(2,3)=\frac{\pi}{648\sqrt{3}}+\frac{1}{648}\ln\frac{1}{3}.$$

于是 $\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)}$

$$=\frac{31\pi}{2592\sqrt{3}}+\frac{13}{162}\ln 2-\frac{1}{32}\ln 3.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\frac{1}{216}\sum_{n=1}^{\infty}\left(\frac{16}{6n-4}-\frac{27}{6n-3}+\frac{12}{6n-2}-\frac{1}{6n-1}\right)$$

$$=\frac{7\pi}{2592\sqrt{3}}+\frac{1}{32}\ln 3-\frac{7}{162}\ln 2.$$

于是 $\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)}-\frac{n^2}{(6n-4)(6n-3)(6n-2)(6n-1)}\right]$$

$$=\frac{\pi}{432\sqrt{3}}+\frac{5}{162}\ln 2-\frac{1}{64}\ln 3.$$

仿此求得

(一) $\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)}$

$$=\frac{1}{384}\sum_{n=1}^{\infty}\left(\frac{49}{8n-7}-\frac{108}{8n-6}+\frac{75}{8n-5}-\frac{16}{8n-4}\right)$$

$$=\frac{31\pi}{768\sqrt{2}}-\frac{67\pi}{3072}+\frac{35}{768}\ln 2$$

$$-\frac{13}{768\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{128}\sum_{n=1}^{\infty}\left(\frac{12}{8n-6}-\frac{25}{8n-5}+\frac{16}{8n-4}-\frac{3}{8n-3}\right)$$

$$=-\frac{11\pi}{512\sqrt{2}}+\frac{17\pi}{1024}-\frac{11}{256}\ln 2$$

$$+\frac{7}{128\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=\frac{1}{384}\sum_{n=1}^{\infty}\left(\frac{25}{8n-5}-\frac{48}{8n-4}+\frac{27}{8n-4}-\frac{4}{8n-2}\right)$$

$$=-\frac{\pi}{1536\sqrt{2}}+\frac{\pi}{1024}+\frac{25}{768}\ln 2$$

$$-\frac{13}{384\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{384}\sum_{n=1}^{\infty}\left(\frac{16}{8n-4}-\frac{27}{8n-3}+\frac{12}{8n-2}-\frac{1}{8n-1}\right)$$

$$=\frac{7\pi}{768\sqrt{2}}-\frac{19\pi}{3072}-\frac{11}{768}\ln 2$$

$$+\frac{13}{768\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=-\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)}\right.$$

$$-\frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)}\Bigg]$$

$$=\frac{95\pi}{6144\sqrt{2}}-\frac{59\pi}{6144}+\frac{17}{768}\ln 2$$

$$-\frac{55}{3072\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=-\frac{\pi}{192\sqrt{2}}+\frac{\pi}{256}-\frac{29}{1536}\ln 2$$

$$+\frac{17}{768\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=-\frac{5\pi}{2048\sqrt{2}}+\frac{11\pi}{6144}+\frac{3}{256}\ln 2$$

$$-\frac{13}{1024\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\Bigg[\frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$-\frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}\Bigg]$$

$$=\frac{127\pi}{30720\sqrt{2}}-\frac{83\pi}{30720}+\frac{21}{2560}\ln 2$$

$$-\frac{41}{5120\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$=-\frac{17\pi}{30720\sqrt{2}}+\frac{13\pi}{30720}-\frac{47}{7680}\ln 2$$

$$+\frac{107}{15360\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n^2}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}\right.$$

$$\left.-\frac{n^2}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}\right].$$

$$=\frac{\pi}{1280\sqrt{2}}-\frac{\pi}{1920}+\frac{11}{4608}\ln 2$$

$$-\frac{23}{9216\sqrt{2}}\ln(\sqrt{2}+1).$$

(二) $$\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-5)(8n-3)(8n-1)}$$

$$=\frac{1}{3072}\sum_{n=1}^{\infty}\left(\frac{49}{8n-7}-\frac{75}{8n-5}+\frac{27}{8n-3}-\frac{1}{8n-1}\right)$$

$$=-\frac{13\pi}{6144\sqrt{2}}+\frac{19\pi}{6144}+\frac{1}{128\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{3072}\sum_{n=1}^{\infty}\left(\frac{25}{8n-5}-\frac{27}{8n-3}+\frac{3}{8n-1}-\frac{1}{8n+1}\right)$$

$$=\frac{\pi}{512\sqrt{2}}-\frac{7\pi}{6144}+\frac{1}{3072\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{3072};$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{n^2}{(8n-7)(8n-5)(8n-3)(8n-1)}\right.$$

$$\left.-\frac{n^2}{(8n-5)(8n-3)(8n-1)(8n+1)}\right]$$

$$=-\frac{25\pi}{49152\sqrt{2}}+\frac{13\pi}{24576}+\frac{23}{24576\sqrt{2}}\ln(\sqrt{2}+1)$$

$$-\frac{1}{24576}.$$

(三) $$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$=\frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{121}{12n-11}-\frac{300}{12n-10}+\frac{243}{12n-9}-\frac{64}{12n-8}\right)$$

$$=-\frac{601\pi}{20736\sqrt{3}}+\frac{485\pi}{20736}+\frac{41}{864}\ln 2-\frac{9}{256}\ln 3$$

$$+\frac{121}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$=\frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{100}{12n-10}-\frac{243}{12n-9}+\frac{192}{12n-8}-\frac{49}{12n-7}\right)$$

$$=\frac{71\pi}{2304\sqrt{3}}-\frac{341\pi}{20736}+\frac{9}{256}\ln 3-\frac{169}{2592}\ln 2$$

$$+\frac{49}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$=\frac{1}{288}\sum_{n=1}^{\infty}\left(\frac{27}{12n-9}-\frac{64}{12n-8}+\frac{49}{12n-7}-\frac{12}{12n-6}\right)$$

$$= -\frac{211\pi}{6912\sqrt{3}} + \frac{125\pi}{6912} + \frac{17}{288}\ln 2 - \frac{5}{768}\ln 3$$

$$-\frac{49}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= \frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{64}{12n-8} - \frac{147}{12n-7} + \frac{108}{12n-6} - \frac{25}{12n-5}\right)$$

$$= \frac{215\pi}{10368\sqrt{3}} - \frac{61\pi}{5184} - \frac{25}{864}\ln 2 - \frac{1}{64}\ln 3$$

$$+\frac{43}{864\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= \frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{49}{12n-7} - \frac{108}{12n-6} + \frac{75}{12n-5} - \frac{16}{12n-4}\right)$$

$$= \frac{47\pi}{10368\sqrt{3}} - \frac{13\pi}{5184} + \frac{13}{864}\ln 2 + \frac{1}{64}\ln 3$$

$$-\frac{31}{864\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$= \frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{36}{12n-6} - \frac{75}{12n-5} + \frac{48}{12n-4} - \frac{9}{12n-3}\right)$$

$$= -\frac{91\pi}{6912\sqrt{3}} + \frac{53\pi}{6912} - \frac{5}{288}\ln 2 - \frac{1}{256}\ln 3$$

$$+\frac{25}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= \frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{25}{12n-5}-\frac{48}{12n-4}+\frac{27}{12n-3}-\frac{4}{12n-2}\right)$$

$$= \frac{5\pi}{768\sqrt{3}}-\frac{77\pi}{20736}+\frac{37}{2592}\ln 2-\frac{1}{256}\ln 3$$

$$-\frac{25}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$= \frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{16}{12n-4}-\frac{27}{12n-3}+\frac{12}{12n-2}-\frac{1}{12n-1}\right)$$

$$= -\frac{49\pi}{20736\sqrt{3}}+\frac{29\pi}{20736}+\frac{1}{256}\ln 3-\frac{5}{864}\ln 2$$

$$-\frac{1}{3456\sqrt{3}}\ln(2+\sqrt{3}).$$

由此顺次推得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(12n-11)(12n-10)(12n-9)(12n-8)}\right.$$

$$\left.-\frac{4n^2}{(12n-5)(12n-4)(12n-3)(12n-2)}\right]$$

$$= 4\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$-4\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$+\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$-4\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= -\frac{13\pi}{324\sqrt{3}} + \frac{77\pi}{2592} - \frac{11}{324}\ln 2$$

$$+ \frac{25}{432\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)}\right.$$

$$\left.-\frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)}\right]$$

$$= -\frac{155\pi}{10368\sqrt{3}} + \frac{413\pi}{41472} + \frac{73}{2592}\ln 2 - \frac{9}{512}\ln 3$$

$$+ \frac{1}{192\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= \frac{159\pi}{10368\sqrt{3}} - \frac{179\pi}{20736} + \frac{1}{96}\ln 3 - \frac{161}{5184}\ln 2$$

$$+ \frac{49}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= -\frac{1063\pi}{82944\sqrt{3}} + \frac{619\pi}{82944} + \frac{19}{864}\ln 2 + \frac{7}{3072}\ln 3$$

$$- \frac{319}{13824\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-8)(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= \frac{7\pi}{1728\sqrt{3}} - \frac{\pi}{432} - \frac{19}{1728}\ln 2 - \frac{1}{128}\ln 3$$

$$+\frac{37}{1728\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-7)(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$=\frac{367\pi}{82944\sqrt{3}}-\frac{211\pi}{82944}+\frac{7}{864}\ln 2+\frac{5}{1024}\ln 3$$

$$-\frac{199}{13824\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-6)(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$=-\frac{17\pi}{3456\sqrt{3}}+\frac{59\pi}{20736}-\frac{41}{5184}\ln 2$$

$$+\frac{25}{3456\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$=\frac{23\pi}{10368\sqrt{3}}-\frac{53\pi}{41472}+\frac{13}{2592}\ln 2-\frac{1}{512}\ln 3$$

$$-\frac{1}{576\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}\right.$$

$$\left.-\frac{4n^2}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}\right]$$

$$=4\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}$$

$$-4\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}$$

$$+\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}$$

$$-4\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n-1)}$$

$$=-\frac{5\pi}{324\sqrt{3}}+\frac{53\pi}{5184}-\frac{1}{108}\ln 2$$

$$+\frac{1}{72\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\Big[\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)}$$

$$-\frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}\Big]$$

$$=-\frac{157\pi}{25920\sqrt{3}}+\frac{257\pi}{69120}+\frac{307}{25920}\ln 2$$

$$-\frac{43}{7680}\ln 3-\frac{31}{17280\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}$$

$$=\frac{467\pi}{82944\sqrt{3}}-\frac{89\pi}{27648}+\frac{5}{3072}\ln 3-\frac{55}{5184}\ln 2$$

$$+\frac{103}{13824\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-4)}$$

$$= -\frac{1399\pi}{414720\sqrt{3}} + \frac{811\pi}{414720} + \frac{19}{2880}\ln 2 + \frac{31}{15360}\ln 3 - \frac{41}{4608\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-8)(12n-7)(12n-6)\cdots(12n-3)}$$

$$= -\frac{31\pi}{414720\sqrt{3}} + \frac{19\pi}{414720} - \frac{11}{2880}\ln 2 - \frac{13}{5120}\ln 3 + \frac{11}{1536\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-7)(12n-6)(12n-5)\cdots(12n-2)}$$

$$= \frac{155\pi}{82944\sqrt{3}} - \frac{149\pi}{138240} + \frac{83}{25920}\ln 2 + \frac{1}{1024}\ln 3 - \frac{299}{69120\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-6)(12n-5)(12n-4)\cdots(12n-1)}$$

$$= -\frac{37\pi}{25920\sqrt{3}} + \frac{19\pi}{23040} - \frac{67}{25920}\ln 2 + \frac{1}{2560}\ln 3 + \frac{31}{17280\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}$$

$$= \frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} - \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}\right]$$

$$= -\frac{4847\pi}{2488320\sqrt{3}} + \frac{959\pi}{829440} + \frac{97}{25920}\ln 2$$

$$-\frac{37}{30720}\ln 3-\frac{71}{46080\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)}$$

$$=\frac{3734\pi}{2488320\sqrt{3}}-\frac{1073\pi}{1244160}-\frac{223}{77760}\ln 2$$

$$-\frac{1}{15360}\ln 3+\frac{113}{41472\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-3)}$$

$$=-\frac{19\pi}{34560\sqrt{3}}+\frac{11\pi}{34560}+\frac{1}{576}\ln 2+\frac{7}{9216}\ln 3$$

$$-\frac{37}{13824\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-8)(12n-7)(12n-6)\cdots(12n-2)}$$

$$=-\frac{403\pi}{1244160\sqrt{3}}+\frac{233\pi}{1244160}-\frac{91}{77760}\ln 2$$

$$-\frac{3}{5120}\ln 3+\frac{397}{207360\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-7)(12n-6)(12n-5)\cdots(12n-1)}$$

$$=\frac{1367\pi}{2488320\sqrt{3}}-\frac{263\pi}{829440}+\frac{5}{5184}\ln 2+\frac{1}{10240}\ln 3$$

$$-\frac{47}{46080\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)}$$

$$=\frac{1}{7}\sum_{n=1}^{\infty}\Big[\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}$$

$$-\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)}\Bigg]$$

$$=-\frac{8581\pi}{17418240\sqrt{3}}+\frac{5023\pi}{17418240}+\frac{257}{272160}\ln 2$$

$$-\frac{1}{6144}\ln 3-\frac{1769}{2903040\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)}$$

$$=\frac{2551\pi}{8709120\sqrt{3}}-\frac{1469\pi}{8709120}-\frac{179}{272160}\ln 2$$

$$-\frac{19}{161280}\ln 3+\frac{1}{1296\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-2)}$$

$$=-\frac{281\pi}{8709120\sqrt{3}}+\frac{163\pi}{8709120}+\frac{113}{272160}\ln 2$$

$$+\frac{31}{161280}\ln 3-\frac{119}{181440\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-8)(12n-7)(12n-6)\cdots(12n-1)}$$

$$=-\frac{2173\pi}{17418240\sqrt{3}}+\frac{251\pi}{3483648}-\frac{83}{272160}\ln 2$$

$$-\frac{1}{10240}\ln 3+\frac{1217}{2903040\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\Bigg[\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)}$$

$$-\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)}\Bigg]$$

$$= -\frac{4561\pi}{46448640\sqrt{3}} + \frac{7961\pi}{139345920} + \frac{109}{544320}\ln 2$$

$$-\frac{29}{5160960}\ln 3 - \frac{4009}{23224320\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}$$

$$= \frac{59\pi}{1451520\sqrt{3}} - \frac{17\pi}{725760} - \frac{73}{544320}\ln 2$$

$$-\frac{5}{129024}\ln 3 + \frac{259}{1451520\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-1)}$$

$$= \frac{179\pi}{15482880\sqrt{3}} - \frac{929\pi}{139345920} + \frac{7}{77760}\ln 2$$

$$+\frac{187}{5160960}\ln 3 - \frac{3121}{23224320\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)}$$

$$= \frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}\right.$$

$$\left.-\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}\right]$$

$$= -\frac{6449\pi}{418037760\sqrt{3}} + \frac{2245\pi}{250822656} + \frac{91}{2449440}\ln 2$$

$$+\frac{19}{5160960}\ln 3 - \frac{8153}{209018880\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)}$$

$$= \frac{193\pi}{59719680\sqrt{3}} - \frac{467\pi}{250822656} - \frac{61}{2449440}\ln 2$$

$$-\frac{43}{5160960}\ln 3+\frac{1453}{41803776\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-1)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)}\right.$$

$$\left.-\frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)}\right]$$

$$=-\frac{13\pi}{6967296\sqrt{3}}+\frac{113\pi}{104509440}+\frac{19}{3061800}\ln 2$$

$$+\frac{31}{25804800}\ln 3-\frac{7709}{1045094400\sqrt{3}}\ln(2+\sqrt{3}).$$

(四) $$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$=\frac{1}{6912}\sum_{n=1}^{\infty}\left(\frac{121}{12n-11}-\frac{243}{12n-9}+\frac{147}{12n-7}-\frac{25}{12n-5}\right)$$

$$=-\frac{17\pi}{18432\sqrt{3}}+\frac{343\pi}{165888}+\frac{9}{2048}\ln 3$$

$$-\frac{1}{27648\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$=\frac{1}{2304}\sum_{n=1}^{\infty}\left(\frac{27}{12n-9}-\frac{49}{12n-7}+\frac{25}{12n-5}-\frac{3}{12n-3}\right)$$

$$=\frac{37\pi}{9216\sqrt{3}}-\frac{59\pi}{27648}-\frac{1}{768}\ln 3$$

$$+\frac{1}{384\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$=\frac{1}{6912}\sum_{n=1}^{\infty}\left(\frac{49}{12n-7}-\frac{75}{12n-5}+\frac{27}{12n-3}-\frac{1}{12n-1}\right)$$

$$=-\frac{41\pi}{18432\sqrt{3}}+\frac{223\pi}{165888}-\frac{1}{2048}\ln 3$$

$$+\frac{25}{27648\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{1}{6912}\sum_{n=1}^{\infty}\left(\frac{25}{12n-5}-\frac{27}{12n-3}+\frac{3}{12n-1}-\frac{1}{12n+1}\right)$$

$$=\frac{7\pi}{18432\sqrt{3}}-\frac{31\pi}{165888}+\frac{1}{2048}\ln 3$$

$$-\frac{23}{27648\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{6912}.$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{4n^2}{(12n-5)(12n-3)(12n-1)(12n+1)}\right]$$

$$=4\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$-4\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$+\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$-4\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{31\pi}{20736}+\frac{23}{3456\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{1728},$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)}\right]$$

$$=-\frac{91\pi}{147456\sqrt{3}}+\frac{697\pi}{1327104}+\frac{35}{49152}\ln 3$$

$$-\frac{73}{221184\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$=\frac{115\pi}{147456\sqrt{3}}-\frac{577\pi}{1327104}-\frac{5}{49152}\ln 3$$

$$+\frac{47}{221184\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-7)(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$=-\frac{\pi}{3072\sqrt{3}}+\frac{127\pi}{663552}-\frac{1}{8192}\ln 3$$

$$+\frac{1}{4608\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{55296},$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n-3)}\right.$$

$$\left.-\frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)}\right]$$

$$=-\frac{103\pi}{737280\sqrt{3}}+\frac{637\pi}{6635520}+\frac{1}{12288}\ln 3$$

$$-\frac{1}{18432\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}$$

$$=\frac{163\pi}{1474560\sqrt{3}}-\frac{277\pi}{4423680}+\frac{1}{491520}\ln 3$$

$$-\frac{1}{2211840\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{552960};$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n+1)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left[\frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}\right.$$

$$\left.-\frac{n^2}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}\right]$$

$$=-\frac{41\pi}{1966080\sqrt{3}}+\frac{421\pi}{31850495}+\frac{13}{1966080}\ln 3$$

$$-\frac{119}{26542080\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{6635520}.$$

例3 求和 $(-1)^{n-1}\dfrac{1}{(5n-4)(5n-3)(5n-2)(5n-1)}$

解 应用分项分式方法求得

$$\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)}$$

$$=\frac{1}{6}\left(\frac{1}{10n-9}-\frac{3}{10n-8}+\frac{3}{10n-7}-\frac{1}{10n-6}\right).$$

于定理11令 $A=1$，$B=3$，$D=3$，$a=1$，$b=2$，$d=3$，$e=4$，$k=10$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}-\frac{3}{10n-8}+\frac{3}{10n-7}-\frac{1}{10n-6}\right)$$

$$=\frac{1}{6}C(1,10)-\frac{1}{4}C(1,5)+\frac{1}{2}C(3,10)-\frac{1}{12}C(2,5)$$

$$+\frac{1}{60}\ln\frac{2^5}{3^3}.$$

根据定理 7，有

$$\frac{1}{6}C(1,10)-\frac{1}{12}C(1,5)=\frac{1}{12}A(1,5);$$

$$\frac{1}{3}C(3,10)-\frac{1}{6}C(1,5)=-\frac{1}{6}A(1,5)+\frac{1}{6}A(3,5)$$

$$+\frac{1}{30}\ln 6;$$

$$\frac{1}{6}C(3,10)-\frac{1}{12}C(2,5)$$

$$=-\frac{1}{12}A(1,5)+\frac{1}{12}A(2,5)+\frac{1}{12}A(3,5)+\frac{1}{60}\ln 6.$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)}$$

$$=-\frac{1}{6}A(1,5)+\frac{1}{12}A(2,5)+\frac{1}{4}A(3,5)+\frac{2}{15}\ln 2$$

$$=0.0417137228\cdots.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{10n-4}-\frac{3}{10n-3}+\frac{3}{10n-2}-\frac{1}{10n-1}\right)$$

$$=-\frac{1}{12}A(1,5)+\frac{1}{3}A(2,5)-\frac{1}{12}A(4,5)-\frac{1}{15}\ln 2$$

$$=0.00034412169\cdots.$$

于是 $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)}\right.$$

$$\left.-\frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)}\right]$$

$$=-\frac{1}{12}A(1,5)-\frac{1}{4}A(2,5)+\frac{1}{4}A(3,5)$$

$$+\frac{1}{12}A(4,5)+\frac{1}{5}\ln 2$$

$$=0.0413696011\cdots.$$

4. $$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{D}{k(n-1)+d}\right.$$

$$\left.-\frac{E}{k(n-1)+e}+\frac{B+E-A-D}{k(n-1)+f}\right]$$

定理12 设 A、B、D、E、b、d、e、f 都是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{D}{k(n-1)+d}\right.$$

$$-\frac{E}{k(n-1)+e}+\frac{B+E-A-D}{k(n-1)+f}\Bigg]$$

$$=AC(a,k)-BC(b,k)+DC(d,k)-EC(e,k)$$

$$+(B+E-A-D)C(f,k)$$

$$+\frac{1}{k}\ln\frac{b^{B}e^{E}}{a^{A}d^{D}f^{B+E-A-D}}.$$

证明 根据定理2，有

$$\sum_{i=1}^{\infty}\Bigg[\frac{A}{k(i-1)+a}-\frac{B}{k(i-1)+b}+\frac{D}{k(i-1)+d}$$

$$-\frac{E}{k(i-1)+e}+\frac{B+E-A-D}{k(i-1)+f}\Bigg]$$

$$=AC(a,k)-BC(b,k)+DC(d,k)-EC(e,k)$$

$$+(B+E-A-D)C(f,k)$$

$$+\frac{1}{k}\ln\frac{(nk-k+a)^{A}(nk-k+d)^{D}(nk-k+f)^{B+E-A-D}}{(nk-k+b)^{B}(nk-k+e)^{E}}$$

$$+\frac{1}{k}\ln\frac{b^{B}e^{E}}{a^{A}d^{D}f^{B+E-A-D}}+A\varepsilon_n(a,k)$$

$$-B\varepsilon_n(b,k)+D\varepsilon_n(d,k)-E\varepsilon_n(e,k)$$

$$+(B+E-A-D)\varepsilon_n(f,k).$$

令 $n\to\infty$ 并取极限，定理即得证。

例1 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)5n}.$$

解 用分项分式方法求得

$$\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)}$$

$$=\frac{1}{24}\left(\frac{1}{10n-9}-\frac{4}{10n-8}+\frac{6}{10n-7}-\frac{4}{10n-6}+\frac{1}{10n-5}\right).$$

于定理12令 $A=1$, $B=4$, $D=6$, $E=4$, $a=1$, $b=2$, $d=3$, $e=4$, $f=5$, $k=10$, 则有

$$\sum_{n=1}^{\infty}\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}-\frac{4}{10n-8}+\frac{6}{10n-7}-\frac{4}{10n-6}+\frac{1}{10n-5}\right)$$

$$=\frac{1}{24}C(1,10)-\frac{1}{12}C(1,5)+\frac{1}{4}C(3,10)-\frac{1}{12}C(2,5)$$

$$+\frac{1}{120}C(1,2)+\frac{1}{240}\ln\frac{2^{12}}{3^{6}5}.$$

根据定理7与6，有

$$\frac{1}{24}C(1,10)-\frac{1}{48}C(1,5)=\frac{1}{48}A(1,5),$$

$$\frac{1}{8}C(3,10)-\frac{1}{16}C(1,5)=-\frac{1}{16}A(1,5)+\frac{1}{16}A(3,5)$$

$$+\frac{1}{80}\ln 6,$$

$$\frac{1}{8}C(3,10)-\frac{1}{16}C(2,5)$$

$$=-\frac{1}{16}A(1,5)+\frac{1}{16}A(2,5)+\frac{1}{16}A(3,5)+\frac{1}{80}\ln 6,$$

$$\frac{1}{120}C(1,2)-\frac{1}{48}C(2,5)$$

$$=-\frac{1}{192}A(1,5)+\frac{1}{96}A(2,5)-\frac{1}{192}A(4,5)$$

$$+\frac{1}{960}\ln\frac{2^{2}}{5}.$$

于是 $\sum_{n=1}^{\infty}\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)}$

$$=-\frac{7}{64}A(1,5)+\frac{7}{96}A(2,5)+\frac{1}{8}A(3,5)$$

$$-\frac{1}{192}A(4,5)+\frac{37}{480}\ln 2-\frac{1}{192}\ln 5$$

$$=0.0083363022\cdots.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)10n}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{1}{10n-4}-\frac{4}{10n-3}+\frac{6}{10n-2}-\frac{4}{10n-1}+\frac{1}{10n}\right)$$

$$=-\frac{5}{192}A(1,5)+\frac{11}{96}A(2,5)+\frac{7}{192}A(4,5)$$

$$-\frac{1}{32}\ln 2-\frac{1}{192}\ln 5$$

$$=0.000033684414\cdots.$$

于是 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)5n}$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)}\right.$$

$$\left.-\frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)10n}\right]$$

$$=-\frac{1}{12}A(1,5)-\frac{1}{24}A(2,5)+\frac{1}{8}A(3,5)-\frac{1}{24}A(4,5)$$

$$+\frac{13}{120}\ln 2$$

$$=0.0083026178\cdots.$$

例2 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)}$$

解 用分项分式方法求得

$$\frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)}$$

$$=\frac{1}{69120}\left(\frac{22}{12n-11}-\frac{105}{12n-7}+\frac{100}{12n-5}-\frac{12}{12n-1}\right.$$

$$\left.-\frac{5}{12n+1}\right).$$

于定理12令 $A=22$，$B=105$，$D=100$，$E=12$，$a=1$，$b=5$，$d=7$，$e=11$，$f=13$，$k=12$，则有

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)}$$

$$=\frac{1}{69120}\sum_{n=1}^{\infty}\left(\frac{22}{12n-11}-\frac{105}{12n-7}+\frac{100}{12n-5}-\frac{12}{12n-1}\right.$$

$$\left.-\frac{5}{12n+1}\right)$$

$$=\frac{11}{34560}C(1,12)-\frac{7}{4608}C(5,12)+\frac{5}{3456}C(7,12)$$

$$-\frac{1}{5760}C(11,12)-\frac{1}{13824}C(13,12)$$

$$+\frac{1}{829440}\ln\frac{5^{105}11^{12}13^{5}}{7^{100}}.$$

于命题2令 $a=1$，$b=1$，$k=12$，则有

$$C(13,12)=C(1,12)+\frac{1}{12}\ln 13-1,$$

$$-\frac{1}{13824}C(13,12)=-\frac{1}{13824}C(1,12)$$

$$-\frac{1}{165888}\ln 13+\frac{1}{13824}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)}$$

$$=\frac{17}{69120}C(1,12)-\frac{7}{4608}C(5,12)+\frac{5}{3456}C(7,12)$$

$$-\frac{1}{5760}C(11,12)+\frac{1}{829440}\ln\frac{5^{105}11^{12}}{7^{100}}+\frac{1}{13824}.$$

根据定理6，有

$$\frac{17}{69120}C(1,12)-\frac{17}{69120}C(5,12)$$

$$=\frac{17\pi}{276480\sqrt{3}}-\frac{1}{829440}\ln 5^{17}$$

$$+\frac{17}{138240\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\frac{11}{8640}C(7,12)-\frac{11}{8640}C(5,12)$$

$$=\frac{11\pi}{34560\sqrt{3}}-\frac{11\pi}{51840}+\frac{1}{103680}\ln\frac{7^{11}}{5^{11}},$$

$$\frac{1}{5760}C(7,12)-\frac{1}{5760}C(11,12)$$

$$=\frac{\pi}{23040\sqrt{3}}+\frac{1}{69120}\ln\frac{7}{11}$$

$$-\frac{1}{11520\sqrt{3}}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)}$$

$$=\frac{13\pi}{30720\sqrt{3}}-\frac{11\pi}{51840}+\frac{1}{27648\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{1}{13824}.$$

例3 求和：

$$\sum_{n=1}^{\infty}\frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}.$$

解 用分项分式方法求得

$$\frac{n^3}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{12288}\left(\frac{343}{8n-7}-\frac{864}{8n-6}+\frac{750}{8n-5}-\frac{256}{8n-4}+\frac{27}{8n-3}\right).$$

于定理12，令 $A=343$，$B=864$，$D=750$，$E=256$，$a=1$，$b=2$，$d=3$，$e=4$，$f=5$，$k=8$，则有

$$\sum_{n=1}^{\infty}\frac{n^3}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=\frac{1}{12288}\sum_{n=1}^{\infty}\left(\frac{343}{8n-7}-\frac{864}{8n-6}+\frac{750}{8n-5}-\frac{256}{8n-4}+\frac{27}{8n-3}\right)$$

$$=\frac{343}{12288}C(1,8)-\frac{9}{256}C(1,4)+\frac{125}{2048}C(3,8)$$

$$-\frac{1}{192}C(1,2)+\frac{9}{4096}C(5,8)+\frac{1}{98304}\ln\frac{2^{1376}}{3^{750}5^{27}}.$$

根据定理 6，有

$$\frac{343}{12288}C(1,8)-\frac{343}{24576}C(1,4)$$

$$=\frac{343\pi}{98304\sqrt{2}}+\frac{343}{49152\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{521}{12288}C(3,8)-\frac{521}{24576}C(1,4)$$

$$=\frac{521\pi}{98304\sqrt{2}}-\frac{521\pi}{98304}+\frac{1}{98304}\ln 3^{521}$$

$$-\frac{521}{49152\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{229}{12288}C(3,8)-\frac{229}{49152}C(1,2)$$

$$=\frac{229\pi}{98304\sqrt{2}}-\frac{229\pi}{196608}+\frac{1}{98304}\ln 3^{229}$$

$$-\frac{229}{49152\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{9}{4096}C(5,8)-\frac{9}{16384}C(1,2)$$

$$=-\frac{9\pi}{32768\sqrt{2}}+\frac{9\pi}{65536}+\frac{1}{32768}\ln 5^{9}$$

$$-\frac{9}{16384\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{n^{3}}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$=\frac{533\pi}{49152\sqrt{2}}-\frac{311\pi}{49152}+\frac{43}{3072}\ln 2$$

$$=\frac{533\pi}{24576\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{n^{3}}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$=\frac{1}{3072}\sum_{n=1}^{\infty}\left(\frac{54}{8n-6}-\frac{125}{8n-5}+\frac{96}{8n-4}-\frac{27}{8n-3}+\frac{2}{8n-2}\right)$$

$$=-\frac{49\pi}{12288\sqrt{2}}+\frac{25\pi}{8192}-\frac{31}{3072}\ln 2$$

$$+\frac{19}{1536\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^3}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{12288}\sum_{n=1}^{\infty}\left(\frac{125}{8n-5}-\frac{256}{8n-4}+\frac{162}{8n-3}-\frac{32}{8n-2}\right.$$

$$\left.+\frac{1}{8n-1}\right)$$

$$=-\frac{19\pi}{49152\sqrt{2}}+\frac{17\pi}{49152}+\frac{17}{3072}\ln 2$$

$$-\frac{143}{24576\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n^3}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}\right.$$

$$\left.-\frac{n^3}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}\right]$$

$$=\frac{243\pi}{81920\sqrt{2}}-\frac{461\pi}{245760}+\frac{37}{7680}\ln 2$$

$$-\frac{521}{122880\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^3}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$=-\frac{59\pi}{81920\sqrt{2}}+\frac{133\pi}{245760}-\frac{1}{320}\ln 2$$

$$+\frac{149}{40960\sqrt{2}}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}\frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}$$

$$=\frac{1}{6}\sum_{n=1}^{\infty}\left[\frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}\right.$$

$$\left.-\frac{n^3}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}\right]$$

$$=\frac{151\pi}{245760\sqrt{2}}-\frac{33\pi}{81920}+\frac{61}{46080}\ln 2$$

$$-\frac{121}{92160\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此求得

(一) $$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{144}\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{9}{12n-9}+\frac{16}{12n-8}-\frac{9}{12n-7}\right.$$

$$\left.+\frac{1}{12n-5}\right)$$

$$=\frac{49\pi}{3456\sqrt{3}}-\frac{\pi}{128}+\frac{1}{128}\ln 3-\frac{1}{36}\ln 2$$

$$+\frac{1}{64\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$=\frac{1}{144}\sum_{n=1}^{\infty}\left(\frac{1}{12n-5}-\frac{9}{12n-3}+\frac{16}{12n-2}-\frac{9}{12n-1}\right.$$

$$\left.+\frac{1}{12n+1}\right)$$

$$=-\frac{5\pi}{1152\sqrt{3}}+\frac{\pi}{128}+\frac{1}{128}\ln 3-\frac{1}{108}\ln 2$$

$$-\frac{1}{64\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{144},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{360}\sum_{n=1}^{\infty}\left(\frac{5}{12n-11}-\frac{12}{12n-10}+\frac{20}{12n-8}-\frac{15}{12n-7}\right.$$

$$\left.+\frac{2}{12n-5}\right)$$

$$=\frac{5\pi}{864\sqrt{3}}-\frac{\pi}{360}-\frac{1}{90}\ln 2+\frac{1}{80\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)}$$

$$=\frac{1}{360}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{12}{12n-4}+\frac{20}{12n-2}-\frac{15}{12n-1}\right.$$

$$\left.+\frac{2}{12n+1}\right)$$

$$=\frac{\pi}{480\sqrt{3}}+\frac{\pi}{360}+\frac{1}{270}\ln 2-\frac{1}{80\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{1}{180},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}$$

$$=\frac{1}{240}\sum_{n=1}^{\infty}\left(\frac{5}{12n-11}-\frac{16}{12n-10}+\frac{15}{12n-9}-\frac{5}{12n-7}\right.$$

$$\left.+\frac{1}{12n-5}\right)$$

$$=-\frac{\pi}{384\sqrt{3}}+\frac{13\pi}{5760}+\frac{1}{180}\ln 2-\frac{1}{128}\ln 3$$

$$+\frac{\sqrt{3}}{320}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{1}{240}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{16}{12n-4}+\frac{15}{12n-3}-\frac{5}{12n-1}\right.$$

$$\left.+\frac{1}{12n+1}\right)$$

$$=\frac{49\pi}{5760\sqrt{3}}-\frac{13\pi}{5760}+\frac{1}{60}\ln 2-\frac{1}{128}\ln 3$$

$$-\frac{\sqrt{3}}{320}\ln(2+\sqrt{3})-\frac{1}{240},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}$$

$$=\frac{1}{360}\sum_{n=1}^{\infty}\left(\frac{10}{12n-11}-\frac{36}{12n-10}+\frac{45}{12n-9}-\frac{20}{12n-8}\right.$$

$$\left.+\frac{1}{12n-5}\right)$$

$$=-\frac{19\pi}{1728\sqrt{3}}+\frac{7\pi}{960}+\frac{1}{45}\ln 2-\frac{1}{64}\ln 3$$

$$+\frac{1}{160\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}$$

$$=\frac{1}{360}\sum_{n=1}^{\infty}\left(\frac{10}{12n-5}-\frac{36}{12n-4}+\frac{45}{12n-3}-\frac{20}{12n-2}\right.$$

$$\left.+\frac{1}{12n+1}\right)$$

$$=\frac{43\pi}{2880\sqrt{3}}-\frac{7\pi}{960}+\frac{4}{135}\ln 2-\frac{1}{64}\ln 3$$

$$-\frac{1}{160\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{360},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{120}\sum_{n=1}^{\infty}\left(\frac{4}{12n-10}-\frac{15}{12n-9}+\frac{20}{12n-8}-\frac{10}{12n-7}\right.$$

$$\left.+\frac{1}{12n-5}\right)$$

$$=\frac{13\pi}{576\sqrt{3}}-\frac{37\pi}{2880}+\frac{1}{64}\ln 3-\frac{2}{45}\ln 2$$

$$+\frac{\sqrt{3}}{160}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$=\frac{1}{120}\sum_{n=1}^{\infty}\left(\frac{4}{12n-4}-\frac{15}{12n-3}+\frac{20}{12n-2}-\frac{10}{12n-1}\right.$$

$$\left.+\frac{1}{12n+1}\right)$$

$$=-\frac{31\pi}{2880\sqrt{3}}+\frac{37\pi}{2880}+\frac{1}{64}\ln 3-\frac{1}{45}\ln 2$$

$$-\frac{\sqrt{3}}{160}\ln(2+\sqrt{3})-\frac{1}{120}.$$

由此求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}\right]$$

$$=\frac{\pi}{54\sqrt{3}}-\frac{\pi}{64}-\frac{1}{54}\ln 2+\frac{1}{32\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{144},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\Big[\frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$+\frac{1}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}\Big]$$

$$=\frac{17\pi}{1728\sqrt{3}}+\frac{1}{64}\ln 3-\frac{1}{27}\ln 2-\frac{1}{144};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\Big[\frac{1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)}$$

$$-\frac{1}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)}\Big]$$

$$=\frac{\pi}{270\sqrt{3}}-\frac{\pi}{180}-\frac{2}{135}\ln 2+\frac{1}{40\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{1}{180},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\Big[\frac{1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)}$$

$$+\frac{1}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)}\Big]$$

$$=\frac{17\pi}{2160\sqrt{3}}-\frac{1}{135}\ln 2-\frac{1}{180};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}\right]$$

$$= -\frac{\pi}{90\sqrt{3}}+\frac{13\pi}{2880}-\frac{1}{90}\ln 2+\frac{\sqrt{3}}{160}\ln(2+\sqrt{3})$$

$$+\frac{1}{240},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.+\frac{1}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}\right]$$

$$= \frac{17\pi}{2880\sqrt{3}}+\frac{1}{45}\ln 2-\frac{1}{64}\ln 3-\frac{1}{240};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}\right]$$

$$= -\frac{7\pi}{270\sqrt{3}}+\frac{7\pi}{480}-\frac{1}{135}\ln 2+\frac{1}{80\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{1}{360},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}\right.$$

$$+\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}\Bigg]$$

$$=\frac{17\pi}{4320\sqrt{3}}+\frac{7}{135}\ln 2-\frac{1}{32}\ln 3-\frac{1}{360};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$-\frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}\Bigg]$$

$$=\frac{\pi}{30\sqrt{3}}-\frac{37\pi}{1440}-\frac{1}{45}\ln 2+\frac{\sqrt{3}}{80}\ln(2+\sqrt{3})+\frac{1}{120},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$+\frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}\Bigg]$$

$$=\frac{17\pi}{1440\sqrt{3}}+\frac{1}{32}\ln 3-\frac{1}{15}\ln 2-\frac{1}{120}.$$

（二） $$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{1728}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{81}{12n-9}+\frac{128}{12n-8}-\frac{63}{12n-7}\right.$$

$$\left.+\frac{5}{12n-5}\right)$$

$$=\frac{365\pi}{41472\sqrt{3}}-\frac{65\pi}{13824}+\frac{3}{512}\ln 3-\frac{1}{54}\ln 2$$

$$+\frac{23}{2304\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$=\frac{1}{1728}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{27}{12n-3}+\frac{32}{12n-2}-\frac{9}{12n-1}\right.$$

$$\left.-\frac{1}{12n+1}\right)$$

$$=-\frac{19\pi}{13824\sqrt{3}}+\frac{11\pi}{13824}+\frac{1}{512}\ln 3-\frac{1}{648}\ln 2$$

$$-\frac{5}{2304\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{1782},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{24}{12n-10}+\frac{32}{12n-8}-\frac{21}{12n-7}\right.$$

$$\left.+\frac{2}{12n-5}\right)$$

$$=\frac{31\pi}{10368\sqrt{3}}-\frac{\pi}{864}-\frac{1}{144}\ln 2$$

$$+\frac{5}{576\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)}$$

$$=\frac{1}{4320}\sum_{n=1}^{\infty}\left(\frac{25}{12n-5}-\frac{48}{12n-4}+\frac{40}{12n-2}-\frac{15}{12n-1}\right.$$

$$\left.-\frac{2}{12n+1}\right)$$

$$=\frac{7\pi}{17280\sqrt{3}}-\frac{\pi}{4320}+\frac{13}{6480}\ln 2$$

$$-\frac{7}{2880\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{2160},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}$$

$$=\frac{1}{576}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{32}{12n-10}+\frac{27}{12n-9}-\frac{7}{12n-7}\right.$$

$$\left.+\frac{1}{12n-5}\right)$$

$$=-\frac{13\pi}{4608\sqrt{3}}+\frac{11\pi}{4608}+\frac{1}{216}\ln 2-\frac{3}{512}\ln 3$$

$$+\frac{17}{2304\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}$$

$$=\frac{1}{2880}\sum_{n=1}^{\infty}\left(\frac{25}{12n-5}-\frac{64}{12n-4}+\frac{45}{12n-3}-\frac{5}{12n-1}\right.$$

$$\left.-\frac{1}{12n+1}\right)$$

$$=\frac{151\pi}{69120\sqrt{3}}-\frac{29\pi}{23040}+\frac{1}{180}\ln 2-\frac{1}{512}\ln 3$$

$$-\frac{31}{11520\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{2280},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}$$

$$=\frac{1}{864}\sum_{n=1}^{\infty}\left(\frac{22}{12n-11}-\frac{72}{12n-10}+\frac{81}{12n-9}-\frac{32}{12n-8}\right.$$

$$\left.+\frac{1}{12n-5}\right)$$

$$= -\frac{179\pi}{20736\sqrt{3}} + \frac{41\pi}{6912} + \frac{7}{432}\ln 2 - \frac{3}{256}\ln 3$$

$$+ \frac{7}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}$$

$$= \frac{1}{4320}\sum_{n=1}^{\infty}\left(\frac{50}{12n-5} - \frac{144}{12n-4} + \frac{135}{12n-3} - \frac{40}{12n-2}\right.$$

$$\left. - \frac{1}{12n+1}\right)$$

$$= \frac{137\pi}{34560\sqrt{3}} - \frac{79\pi}{34560} + \frac{59}{6480}\ln 2 - \frac{1}{256}\ln 3$$

$$- \frac{17}{5760\sqrt{3}}\ln(2+\sqrt{3}) + \frac{1}{4320},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$= \frac{1}{288}\sum_{n=1}^{\infty}\left(\frac{8}{12n-10} - \frac{27}{12n-9} + \frac{32}{12n-8} - \frac{14}{12n-7}\right.$$

$$\left. + \frac{1}{12n-5}\right)$$

$$= \frac{101\pi}{6912\sqrt{3}} - \frac{19\pi}{2304} + \frac{3}{256}\ln 3 - \frac{13}{432}\ln 2$$

$$+ \frac{13}{1152\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$= \frac{1}{1440}\sum_{n=1}^{\infty}\left(\frac{16}{12n-4} - \frac{45}{12n-3} + \frac{40}{12n-2} - \frac{10}{12n-1}\right.$$

$$\left. - \frac{1}{12n+1}\right)$$

$$= -\frac{109\pi}{34560\sqrt{3}}+\frac{7\pi}{3840}+\frac{1}{256}\ln 3-\frac{11}{2160}\ln 2$$

$$-\frac{11}{5760\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{1440}.$$

由此求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$=\frac{\pi}{162\sqrt{3}}-\frac{11\pi}{3456}-\frac{1}{162}\ln 2$$

$$+\frac{5}{576\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{864},$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}\right.$$

$$\left.+\frac{2n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}\right]$$

$$=\frac{7\pi}{10368\sqrt{3}}+\frac{1}{128}\ln 3-\frac{1}{81}\ln 2+\frac{1}{864};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)}\right]$$

$$=-\frac{\pi}{1620\sqrt{3}}+\frac{\pi}{1080}-\frac{11}{1620}\ln 2$$

$$+\frac{7}{720\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{1080},$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)}\right.$$

$$\left.+\frac{2n}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)}\right]$$

$$=\frac{13\pi}{12960\sqrt{3}}+\frac{1}{810}\ln 2+\frac{1}{1080};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}\right]$$

$$=-\frac{\pi}{135\sqrt{3}}+\frac{29\pi}{5760}-\frac{1}{135}\ln 2$$

$$+\frac{31}{2880\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{1440},$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}\right.$$

$$\left.+\frac{2n}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}\right]$$

$$=\frac{23\pi}{17280\sqrt{3}}+\frac{2}{135}\ln 2-\frac{1}{128}\ln 3+\frac{1}{1440};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}\right]$$

$$=-\frac{23\pi}{1620\sqrt{3}}+\frac{79\pi}{8640}-\frac{13}{1620}\ln 2$$

$$+\frac{17}{1440\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{2160},$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}\right.$$

$$\left.+\frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}\right]$$

$$=\frac{43\pi}{25920\sqrt{3}}+\frac{23}{810}\ln 2-\frac{1}{64}\ln 3+\frac{1}{2160};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}\right.$$

$$-\frac{2n}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}\Big]$$

$$=\frac{7\pi}{540\sqrt{3}}-\frac{7\pi}{960}-\frac{1}{180}\ln 2+\frac{11}{1440\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{1}{720},$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\Big[\frac{2n-1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$+\frac{2n}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}\Big]$$

$$=\frac{\pi}{2880\sqrt{3}}+\frac{1}{64}\ln 3-\frac{7}{270}\ln 2+\frac{1}{720}.$$

(三) $$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-1)}$$

$$=\frac{1}{360}\sum_{n=1}^{\infty}\Big(\frac{10}{8n-7}-\frac{36}{8n-6}+\frac{45}{8n-5}-\frac{20}{8n-4}$$

$$+\frac{1}{8n-1}\Big)$$

$$=\frac{3\pi}{160\sqrt{2}}-\frac{\pi}{80}+\frac{19}{720}\ln 2-\frac{17}{720\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{360}\sum_{n=1}^{\infty}\Big(\frac{1}{8n-7}-\frac{20}{8n-4}+\frac{45}{8n-3}-\frac{36}{8n-2}$$

$$+\frac{10}{8n-1}\Big)$$

$$= -\frac{3\pi}{160\sqrt{2}} + \frac{\pi}{80} + \frac{19}{720}\ln 2$$

$$- \frac{17}{720\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)}$$

$$= \frac{1}{240}\sum_{n=1}^{\infty}\left(\frac{5}{8n-7} - \frac{16}{8n-6} + \frac{15}{8n-5} - \frac{5}{8n-3} + \frac{1}{8n-1}\right)$$

$$= \frac{\pi}{80\sqrt{2}} - \frac{\pi}{120} + \frac{1}{120}\ln 2 - \frac{1}{240\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)}$$

$$= \frac{1}{144}\sum_{n=1}^{\infty}\left(\frac{1}{8n-7} - \frac{9}{8n-5} + \frac{16}{8n-4} - \frac{9}{8n-3} + \frac{1}{8n-1}\right)$$

$$= -\frac{1}{36}\ln 2 + \frac{5}{144\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(8n-3)(8n-2)(8n-1)}$$

$$= \frac{1}{240}\sum_{n=1}^{\infty}\left(\frac{1}{8n-7} - \frac{5}{8n-5} + \frac{15}{8n-3} - \frac{16}{8n-2} + \frac{5}{8n+1}\right)$$

$$= -\frac{\pi}{80\sqrt{2}} + \frac{\pi}{120} + \frac{1}{120}\ln 2$$

$$- \frac{1}{240\sqrt{2}}\ln(\sqrt{2}+1).$$

(四) $$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-1)}$$

$$=\frac{1}{2880}\sum_{n=1}^{\infty}\left(\frac{70}{8n-7}-\frac{216}{8n-6}+\frac{225}{8n-5}-\frac{80}{8n-4}+\frac{1}{8n-1}\right)$$

$$=\frac{49\pi}{3840\sqrt{2}}-\frac{31\pi}{3840}+\frac{47}{2880}\ln 2-\frac{77}{5760\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{2880}\sum_{n=1}^{\infty}\left(\frac{7}{8n-7}-\frac{80}{8n-4}+\frac{135}{8n-3}-\frac{72}{8n-2}+\frac{10}{8n-1}\right)$$

$$=-\frac{23\pi}{3840\sqrt{2}}+\frac{17\pi}{3840}+\frac{29}{2880}\ln 2-\frac{59}{5760\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)}$$

$$=\frac{1}{1920}\sum_{n=1}^{\infty}\left(\frac{35}{8n-7}-\frac{96}{8n-6}+\frac{75}{8n-5}-\frac{15}{8n-3}+\frac{1}{8n-1}\right)$$

$$=\frac{31\pi}{3840\sqrt{2}}-\frac{19\pi}{3840}+\frac{1}{160}\ln 2-\frac{1}{320\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)}$$

$$=\frac{1}{1152}\sum_{n=1}^{\infty}\left(\frac{7}{8n-7}-\frac{45}{8n-5}+\frac{64}{8n-4}-\frac{27}{8n-3}+\frac{1}{8n-1}\right)$$

$$=-\frac{\pi}{768\sqrt{2}}+\frac{\pi}{768}-\frac{1}{72}\ln 2+\frac{5}{288\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{1920}\sum_{n=1}^{\infty}\left(\frac{7}{8n-7}-\frac{25}{8n-5}+\frac{45}{8n-3}-\frac{32}{8n-2}+\frac{5}{8n-1}\right)$$

$$=-\frac{17\pi}{3840\sqrt{2}}+\frac{13\pi}{3840}+\frac{1}{480}\ln 2-\frac{1}{960\sqrt{2}}\ln(\sqrt{2}+1).$$

5. $$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{D}{k(n-1)+d}-\frac{E}{k(n-1)+e}+\frac{F}{k(n-1)+f}-\frac{A+D+F-B-E}{k(n-1)+g}\right]$$

仿照以上几个定理的证法，即可证明下面这个定理.

定理13 设 A、B、D、E、F、b、d、e、f、g 都是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(n-1)+b}+\frac{D}{k(n-1)+d}\right.$$

$$\left.-\frac{E}{k(n-1)+e}+\frac{F}{k(n-1)+f}-\frac{A+D+F-B-E}{k(n-1)+g}\right]$$

$$=AC(a,k)-BC(b,k)+DC(d,k)-EC(e,k)$$

$$+FC(f,k)-(A+D+F-B-E)C(g,k)$$

$$+\frac{1}{k}\ln\frac{b^{B}e^{E}g^{A+D+F-B-E}}{a^{A}d^{D}f^{F}}.$$

例1 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)}$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}.$$

解 用分项分式方法求得

$$\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{720}\left(\frac{5}{12n-11}-\frac{24}{12n-10}+\frac{45}{11n-9}-\frac{40}{12n-8}\right.$$

$$\left.+\frac{15}{12n-7}-\frac{1}{12n-5}\right).$$

于定理13令$A=5$，$B=24$，$D=45$，$E=40$，$F=15$，$a=1$，$b=2$，$d=3$，$e=4$，$f=5$，$g=7$，$k=12$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$=\frac{1}{720}\sum_{n=1}^{\infty}\left(\frac{5}{12n-11}-\frac{24}{12n-10}+\frac{45}{12n-9}-\frac{40}{12n-8}\right.$$

$$+\frac{15}{12n-7}-\frac{1}{12n-1}\Big)$$

$$=\frac{1}{144}C(1,12)-\frac{1}{60}C(1,6)+\frac{1}{48}C(1,4)-\frac{1}{72}C(1,3)$$

$$+\frac{1}{48}C(5,12)-\frac{1}{720}C(7,12)+\frac{1}{8640}\ln\frac{2^{104}7}{3^{45}5^{15}}.$$

根据定理6，有

$$\frac{1}{144}C(1,12)-\frac{1}{288}C(1,6)$$

$$=\frac{\pi}{1728}+\frac{1}{576\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\frac{19}{2160}C(1,4)-\frac{19}{1440}C(1,6)$$

$$=-\frac{19\pi}{5760\sqrt{3}}+\frac{19\pi}{17280}-\frac{1}{17280}\ln 3^{19},$$

$$\frac{13}{1080}C(1,4)-\frac{13}{1440}C(1,3)$$

$$=-\frac{13\pi}{8640\sqrt{3}}+\frac{13\pi}{8640}+\frac{1}{8640}\ln\frac{2^{26}}{3^{13}},$$

$$\frac{7}{360}C(5,12)-\frac{7}{1440}C(1,3)$$

$$=-\frac{7\pi}{2160\sqrt{3}}+\frac{7\pi}{4320}+\frac{1}{4320}\ln 2^{7}\cdot 5^{7}$$

$$-\frac{7}{1440\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\frac{1}{720}C(5,12)-\frac{1}{720}C(7,12)$$

$$=-\frac{\pi}{2880\sqrt{3}}+\frac{\pi}{4320}+\frac{1}{8640}\ln\frac{5}{7}.$$

于是

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$=-\frac{29\pi}{3456\sqrt{3}}+\frac{29\pi}{5760}+\frac{1}{60}\ \ln 2-\frac{1}{128}\ \ln 3$$

$$-\frac{1}{320\sqrt{3}}\ \ln(2+\sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12-1)(12+1)}$$

$$=\frac{1}{720}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{24}{12n-4}+\frac{45}{12n-3}-\frac{40}{12n-2}\right.$$

$$\left.+\frac{15}{12n-1}-\frac{1}{12n+1}\right)$$

$$=\frac{37\pi}{5760\sqrt{3}}-\frac{29\pi}{5760}+\frac{7}{540}\ \ln 2-\frac{1}{128}\ \ln 3$$

$$+\frac{1}{320\sqrt{3}}\ \ln(2+\sqrt{3})+\frac{1}{720}.$$

于是 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)}$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}\right.$$

$$\left.-\frac{1}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}\right]$$

$$=-\frac{2\pi}{135\sqrt{3}}+\frac{29\pi}{2880}+\frac{1}{270}\ \ln 2$$

$$-\frac{1}{160\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{720},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{2(1n-11)(12n-10)(12n-9)\cdots(12n-5)}\right.$$

$$\left.+\frac{1}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}\right]$$

$$=-\frac{17\pi}{8640\sqrt{3}}+\frac{4}{135}\ln 2-\frac{1}{64}\ln 3+\frac{1}{720}.$$

例2 求和:

$$\sum_{n=1}^{\infty}\frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}.$$

解 用分项分式方法求得

$$\frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{1}{491520}\left(\frac{2401}{8n-7}-\frac{6480}{8n-6}+\frac{6250}{8n-5}-\frac{2560}{8n-4}\right.$$

$$\left.+\frac{405}{8n-3}-\frac{16}{8n-2}\right).$$

于定理13令 $A=2401$, $B=6480$, $D=6250$, $E=2560$, $=405$, $a=1$, $b=2$, $d=3$, $e=4$, $f=5$, $g=6$, $k=8$, 则有

$$\sum_{n=1}^{\infty}\frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{1}{491520}\sum_{n=1}^{\infty}\left(\frac{2401}{8n-7}-\frac{6480}{8n-6}+\frac{6250}{8n-5}-\frac{2560}{8n-4}\right.$$

$$+\frac{405}{8n-3}-\frac{16}{8n-2}\Big)$$

$$=\frac{2401}{491520}C(1,8)-\frac{27}{4096}C(1,4)+\frac{625}{49152}C(3,8)$$

$$-\frac{1}{768}C(1,2)+\frac{27}{32768}C(5,8)-\frac{1}{61440}C(3,4)$$

$$+\frac{1}{3932160}\ln\frac{2^{11616}}{3^{6234}5^{405}}.$$

根据定理6，有

$$\frac{2401}{491520}C(1,8)-\frac{2401}{983040}C(1,4)$$

$$=\frac{2401\pi}{3932160\sqrt{2}}+\frac{2401}{1966080\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{4079}{491520}C(3,8)-\frac{4079}{983040}C(1,4)$$

$$=\frac{4079\pi}{3932160\sqrt{2}}-\frac{4079\pi}{3932160}+\frac{1}{3932160}\ln 3^{4079}$$

$$-\frac{4079}{1966080\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{2171}{491520}C(3,8)-\frac{2171}{1966080}C(1,2)$$

$$=\frac{2171\pi}{3932160\sqrt{2}}-\frac{2171\pi}{7864320}+\frac{1}{3932160}\ln 3^{2171}$$

$$-\frac{2171}{1966080\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{389}{491520}C(5,8)-\frac{389}{1966080}C(1,2)$$

$$=-\frac{389\pi}{3932160\sqrt{2}}+\frac{389\pi}{7864320}+\frac{1}{3932160}\ln 5^{389}$$

$$-\frac{389}{1966080\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\frac{1}{30720}C(5,8)-\frac{1}{61440}C(3,4)$$

$$=-\frac{\pi}{245760\sqrt{2}}+\frac{\pi}{245760}+\frac{1}{245760}\ln\frac{5}{3}$$

$$-\frac{1}{122880\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$=\frac{4123\pi}{1966080\sqrt{2}}-\frac{2477\pi}{1966080}+\frac{121}{40960}\ln 2$$

$$-\frac{709}{327680\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{n^4}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$=\frac{1}{491520}\sum_{n=1}^{\infty}\left(\frac{1296}{8n-6}-\frac{3125}{8n-5}+\frac{2560}{8n-4}-\frac{810}{8n-3}\right.$$

$$\left.+\frac{80}{8n-2}-\frac{1}{8n-1}\right)$$

$$=-\frac{1157\pi}{1966080\sqrt{2}}+\frac{883\pi}{1966080}-\frac{203}{122880}\ln 2$$

$$+\frac{1967}{983040\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}$$

$$=\frac{11\pi}{24576\sqrt{2}}-\frac{7\pi}{24576}+\frac{283}{368640}\ln 2$$

$$-\frac{2047}{2949120\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此求得

(一) $$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}$$

$$=\frac{1}{1728}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{48}{12n-10}+\frac{81}{12n-9}-\frac{64}{12n-8}\right.$$

$$\left.+\frac{21}{12n-7}-\frac{1}{12n-5}\right)$$

$$=-\frac{241\pi}{41472\sqrt{3}}+\frac{49\pi}{13824}+\frac{5}{432}\ln 2$$

$$-\frac{3}{512}\ln 3-\frac{1}{768\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)\cdots(12n+1)}$$

$$=\frac{1}{8640}\sum_{n=1}^{\infty}\left(\frac{25}{12n-5}-\frac{96}{12n-4}+\frac{135}{12n-3}-\frac{80}{12n-2}\right.$$

$$\left.+\frac{15}{12n-1}+\frac{1}{12n+1}\right)$$

$$=\frac{41\pi}{23040\sqrt{3}}-\frac{71\pi}{69120}+\frac{23}{6480}\ln 2-\frac{1}{512}\ln 3$$

$$-\frac{1}{3840\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{8640}.$$

由此求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)}\right.$$

$$\left.-\frac{2n}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}\right]$$

$$=-\frac{11\pi}{1620\sqrt{3}}+\frac{71\pi}{17280}-\frac{1}{1620}\ln 2$$

$$+\frac{1}{960\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{4320},$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)}\right.$$

$$\left.+\frac{2n}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}\right]$$

$$=\frac{17\pi}{51840\sqrt{3}}+\frac{11}{810}\ln 2-\frac{1}{128}\ln 3-\frac{1}{4320}.$$

(二) $$\sum_{n=1}^{\infty}\frac{n^2}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)}$$

$$=\frac{1}{20736}\sum_{n=1}^{\infty}\left(\frac{121}{12n-11}-\frac{480}{12n-10}+\frac{729}{12n-9}-\frac{512}{12n-8}\right.$$

$$\left.+\frac{147}{12n-7}-\frac{5}{12n-5}\right)$$

$$=-\frac{2045\pi}{497664\sqrt{3}}+\frac{425\pi}{165888}+\frac{7}{864}\ln 2-\frac{27}{6144}\ln 3$$

$$-\frac{7}{27648\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}$$

$$=\frac{1}{103680}\sum_{n=1}^{\infty}\left(\frac{125}{12n-5}-\frac{384}{12n-4}+\frac{405}{12n-3}-\frac{160}{12n-2}\right.$$

$$\left.+\frac{15}{12n-1}+\frac{1}{12n+1}\right)$$

$$=\frac{397\pi}{829440\sqrt{3}}-\frac{229\pi}{829440}+\frac{41}{38880}\ln 2-\frac{1}{2048}\ln 3$$

$$-\frac{37}{138240\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{103680}.$$

由此求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)}\right.$$

$$\left.-\frac{4n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}\right]$$

$$=-\frac{17\pi}{4860\sqrt{3}}+\frac{229\pi}{103680}-\frac{7}{4860}\ln 2$$

$$+\frac{37}{17280\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{25920},$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)}\right.$$

$$\left.+\frac{4n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)}\right]$$

$$=\frac{103\pi}{311040\sqrt{3}}+\frac{17}{2430}\ln 2-\frac{1}{256}\ln 3+\frac{1}{25920}.$$

(三)

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-7)(12n-5)(12n-1)(12n+1)}$$

$$=\frac{1}{190080}\sum_{n=1}^{\infty}\left(\frac{66}{12n-11}-\frac{128}{12n-10}+\frac{165}{12n-7}-\frac{132}{12n-5}\right.$$

$$\left.+\frac{44}{12n-1}-\frac{15}{12n+1}\right)$$

$$=-\frac{19\pi}{69120\sqrt{3}}+\frac{19\pi}{142560}+\frac{1}{17820}\ln 2$$

$$+\frac{31}{380160\sqrt{3}}\ln(2+\sqrt{3})+\frac{1}{12672},$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-7)(12n-5)(12n-1)(12n+1)}$$

$$=\frac{1}{2280960}\sum_{n=1}^{\infty}\left(\frac{726}{12n-11}-\frac{1280}{12n-10}+\frac{1155}{12n-7}\right.$$

$$\left.-\frac{660}{12n-5}+\frac{44}{12n-1}+\frac{15}{12n+1}\right)$$

$$=-\frac{109\pi}{829440\sqrt{3}}+\frac{157\pi}{1710720}+\frac{1}{21384}\ln 2$$

$$+\frac{29}{912384\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{152064},$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)(12n-1)}$$

$$=\frac{1}{17280}\sum_{n=1}^{\infty}\left(\frac{36}{12n-11}-\frac{128}{12n-10}+\frac{135}{12n-9}-\frac{60}{12n-7}\right.$$

$$\left.+\frac{18}{12n-5}-\frac{1}{12n-1}\right)$$

$$=-\frac{13\pi}{138240\sqrt{3}}+\frac{53\pi}{414720}+\frac{1}{1620}\ln 2-\frac{1}{1024}\ln 3$$

$$+\frac{77}{69120\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)(12n-1)}$$

$$=\frac{1}{207360}\sum_{n=1}^{\infty}\left(\frac{396}{12n-11}-\frac{1280}{12n-10}+\frac{1215}{12n-9}-\frac{420}{12n-7}\right.$$

$$\left.+\frac{90}{12n-5}-\frac{1}{12n-1}\right)$$

$$=-\frac{373\pi}{1658880\sqrt{3}}+\frac{989\pi}{4976640}+\frac{1}{1944}\ln 2-\frac{3}{4096}\ln 3$$

$$+\frac{145}{165888\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)(12n-1)}$$

$$=\frac{1}{20160}\sum_{n=1}^{\infty}\left(\frac{56}{12n-11}-\frac{224}{12n-10}+\frac{315}{12n-9}-\frac{160}{12n-8}\right.$$

$$+\frac{14}{12n-5}-\frac{1}{12n-1}\Big)$$

$$=-\frac{619\pi}{483840\sqrt{3}}+\frac{401\pi}{483840}+\frac{11}{3780}\ln 2-\frac{1}{512}\ln 3$$

$$+\frac{41}{80640\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)(12n-1)}$$

$$=\frac{1}{241920}\sum_{n=1}^{\infty}\Big(\frac{616}{12n-11}-\frac{2240}{12n-10}+\frac{2835}{12n-9}-\frac{1280}{12n-8}$$

$$+\frac{70}{12n-5}-\frac{1}{12n-1}\Big)$$

$$=-\frac{5939\pi}{5806080\sqrt{3}}+\frac{3929\pi}{5806080}+\frac{19}{9072}\ln 2-\frac{3}{2048}\ln 3$$

$$+\frac{109}{193536\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$=\frac{1}{30240}\sum_{n=1}^{\infty}\Big(\frac{1}{12n-11}-\frac{210}{12n-5}+\frac{720}{12n-4}-\frac{945}{12n-3}$$

$$+\frac{560}{12n-2}-\frac{126}{12n-1}\Big)$$

$$=-\frac{883\pi}{241920\sqrt{3}}+\frac{1619\pi}{725760}+\frac{1}{256}\ln 3-\frac{17}{2268}\ln 2$$

$$+\frac{17}{24192\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{n}{(12n-11)(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$= \frac{1}{362880} \sum_{n=1}^{\infty} \left(\frac{11}{12n-11} - \frac{1050}{12n-5} + \frac{2880}{12n-4} - \frac{2835}{12n-3} \right.$$

$$\left. + \frac{1120}{12n-2} - \frac{126}{12n-1} \right)$$

$$= -\frac{2993\pi}{2903040\sqrt{3}} + \frac{5209\pi}{8709120} + \frac{1}{1024} \ln 3 - \frac{61}{27216} \ln 2$$

$$+ \frac{187}{290304\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-1)}$$

$$= \frac{1}{30240} \sum_{n=1}^{\infty} \left(\frac{126}{12n-11} - \frac{560}{12n-10} + \frac{945}{12n-9} - \frac{720}{12n-8} \right.$$

$$\left. + \frac{210}{12n-7} - \frac{1}{12n-1} \right)$$

$$= -\frac{883\pi}{241920\sqrt{3}} + \frac{1619\pi}{725760} + \frac{17}{2268} \ln 2 - \frac{1}{256} \ln 3$$

$$- \frac{17}{24192\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-1)}$$

$$= \frac{1}{362880} \sum_{n=1}^{\infty} \left(\frac{1386}{12n-11} - \frac{5600}{12n-10} + \frac{8505}{12n-9} - \frac{5760}{12n-8} \right.$$

$$\left. + \frac{1470}{12n-7} - \frac{1}{12n-1} \right)$$

$$= -\frac{7603\pi}{2903040\sqrt{3}} + \frac{14219\pi}{8709120} + \frac{143}{27216} \ln 2 - \frac{3}{1024} \ln 3$$

$$- \frac{17}{290304\sqrt{3}} \ln(2+\sqrt{3}).$$

(四) $$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{13440}\sum_{n=1}^{\infty}\left(\frac{35}{8n-7}-\frac{128}{8n-6}+\frac{140}{8n-5}-\frac{70}{8n-3}\right.$$

$$\left.+\frac{28}{8n-1}-\frac{5}{8n+1}\right)$$

$$=\frac{53\pi}{26880\sqrt{2}}-\frac{\pi}{640}+\frac{1}{840}\ln 2$$

$$-\frac{1}{4480\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{2688},$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{107520}\sum_{n=1}^{\infty}\left(\frac{245}{8n-7}-\frac{768}{8n-6}+\frac{700}{8n-5}-\frac{210}{8n-3}\right.$$

$$\left.+\frac{28}{8n-1}+\frac{5}{8n+1}\right)$$

$$=\frac{283\pi}{215040\sqrt{2}}-\frac{13\pi}{15360}+\frac{1}{1120}\ln 2$$

$$-\frac{53}{107520\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{21504},$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{5760}\sum_{n=1}^{\infty}\left(\frac{5}{8n-7}-\frac{60}{8n-5}+\frac{128}{8n-4}-\frac{90}{8n-3}\right.$$

$$\left.+\frac{20}{8n-1}-\frac{3}{8n+1}\right)$$

$$=\frac{\pi}{3840\sqrt{2}}-\frac{\pi}{1920}-\frac{1}{180}\ln 2+\frac{43}{5760\sqrt{2}}\ln(\sqrt{2}+1)$$

$$+\frac{1}{1920}.$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)(8n+1)}$$

$$=\frac{1}{46080}\sum_{n=1}^{\infty}\left(\frac{35}{8n-7}-\frac{300}{8n-5}+\frac{512}{8n-4}-\frac{270}{8n-3}\right.$$

$$\left.+\frac{20}{8n-1}+\frac{3}{8n+1}\right)$$

$$=-\frac{\pi}{30720\sqrt{2}}+\frac{\pi}{15360}-\frac{1}{360}\ln 2$$

$$+\frac{157}{46080\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{15360},$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{240}\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{4}{8n-6}+\frac{5}{8n-5}-\frac{5}{8n-3}\right.$$

$$\left.+\frac{4}{8n-2}-\frac{1}{8n-1}\right)$$

$$=\frac{\pi}{160\sqrt{2}}-\frac{\pi}{240},$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-2)(8n-1)}$$

$$=\frac{1}{1920}\sum_{n=1}^{\infty}\left(\frac{7}{8n-7}-\frac{24}{8n-6}+\frac{25}{8n-5}-\frac{15}{8n-3}+\frac{8}{8n-2}\right.$$

$$\left.-\frac{1}{8n-1}\right)$$

$$=\frac{\pi}{320\sqrt{2}}-\frac{\pi}{480}+\frac{1}{960}\ln 2$$

$$-\frac{1}{1920\sqrt{2}}\ln(\sqrt{2}+1).$$

在三、中已经看到，当 $k=1,2,3,4,6$ 时，容易求得

$A(a,k)$的有限表达式．当 $k=5$ 或 8 时，也可以求得 $A(a,k)$的有限表达式,但计算过程很复杂．对于任意的 $A(a,k)$，难以求得其有限的表达式．应用上述的定理可推得收敛迅速的级数来表示 $A(a,k)$，只要计算所得级数的前若干项之和，即得到 $A(a,k)$ 的近似值，并达到很高的精确度．

例3 求 $A(1,18)$．

解 用分项分式方法求得

$$\frac{1}{(36n-35)\cdots(36n-33)(36n-30)(36n-27)(36n-24)}$$

$$=\frac{1}{498960}\left(\frac{567}{36n-35}-\frac{1782}{36n-34}+\frac{1540}{36n-33}-\frac{462}{36n-30}\right.$$
$$\left.+\frac{165}{36n-27}-\frac{28}{36n-24}\right).$$

于定理13令 $A=567$，$B=1782$，$D=1540$，$E=462$，$F=165$，$a=1$，$b=2$，$d=3$，$e=6$，$f=9$，$g=12$，$k=36$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(36n-35)\cdots(36n-33)(36n-30)(12n-27)(36n-34)}$$

$$=\frac{1}{498960}\sum_{n=1}^{\infty}\left(\frac{567}{36n-35}-\frac{1782}{36n-34}+\frac{1540}{36n-33}\right.$$
$$\left.-\frac{462}{36n-30}+\frac{165}{36n-27}-\frac{28}{36n-24}\right)$$

$$=\frac{1}{880}C(1,36)-\frac{1}{560}C(1,18)+\frac{1}{972}C(1,12)$$
$$-\frac{1}{6480}C(1,6)+\frac{1}{27216}C(1,4)-\frac{1}{213840}C(1,3)$$
$$+\frac{1}{17962560}\ln\frac{2^{2300}}{3^{1880}}.$$

根据定理5与6，有

$$\frac{1}{160380}C(1,4)-\frac{1}{213840}C(1,3)$$

$$=-\frac{\pi}{1283040\sqrt{3}}+\frac{\pi}{1283040}+\frac{1}{1283040}\ln\frac{2^{3}}{3},$$

$$\frac{137}{4490640}C(1,4)-\frac{137}{2993760}C(1,6)$$

$$=-\frac{137\pi}{11975040\sqrt{3}}+\frac{137\pi}{35925120}$$

$$-\frac{1}{35925120}\ln 3^{137},$$

$$\frac{1}{972}C(1,12)-\frac{65}{598952}C(1,6)$$

$$=\frac{1}{1232}C(1,12)+\frac{65}{299376}C(1,12)-\frac{65}{598952}C(1,6)$$

$$=\frac{1}{1232}C(1,12)+\frac{65\pi}{3592512}$$

$$+\frac{65}{1197504\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\frac{1}{880}C(1,36)-\frac{1}{560}C(1,18)$$

$$=\frac{1}{880}C(1,36)-\frac{1}{1760}C(1,18)-\frac{3}{2464}C(1,18)$$

$$=\frac{1}{1760}A(1,18)-\frac{3}{2464}C(1,18).$$

于是

$$\sum_{n=1}^{\infty}\frac{1}{(36n-35)\cdots(36n-33)(36n-30)(36n-27)(36n-24)}$$

$$=\frac{1}{1760}A(1,18)-\frac{3}{2464}C(1,18)+\frac{1}{1232}C(1,12)$$

$$+\frac{163\pi}{7185024}-\frac{439\pi}{35925120\sqrt{3}}+\frac{97}{748440}\ln 2$$

$$-\frac{65}{798336}\ln 3+\frac{65}{1197504\sqrt{3}}\ln(2+\sqrt{3}).$$

仿此推得

$$\sum_{n=1}^{\infty}\frac{1}{(36n-34)(36n-33)(36n-30)(36n-27)\cdots(36n-21)}$$

$$=\frac{1}{884520}\sum_{n=1}^{\infty}\left(\frac{243}{36n-34}-\frac{455}{36n-33}+\frac{455}{36n-30}\right.$$

$$\left.-\frac{390}{36n-27}+\frac{182}{36n-24}-\frac{35}{36n-21}\right)$$

$$=\frac{1}{7280}C(1,18)-\frac{1}{10920}C(1,12)-\frac{221\pi}{15921360}$$

$$+\frac{127\pi}{7960680\sqrt{3}}+\frac{23}{884520}\ln 3$$

$$-\frac{311}{7960680}\ln 2-\frac{59}{3538080\sqrt{3}}\ln(2+\sqrt{3}).$$

由此解出 $A(1,18)$即得

$A(1,18)$

$$=\sum_{n=1}^{\infty}\frac{1760}{(36n-35)\cdots(36n-33)(36n-30)(36n-27)(36n-24)}$$

$$+\sum_{n=1}^{\infty}\frac{15600}{(36n-34)(36n-33)(36n-30)(36n-27)\cdots(36n-21)}$$

$$-\frac{1547\pi}{6804\sqrt{3}}+\frac{3605\pi}{20412}+\frac{1946}{5103}\ln 2-\frac{595}{2268}\ln 3$$

$$+\frac{560}{3402\sqrt{3}}\ln(2+\sqrt{3}).$$

计算右端两级数之和的前10位小数，得

$$\sum_{n=1}^{6}\frac{1760}{(36n-35)(36n-34)(36n-33)(36n-30)\cdots(36n-24)}$$
$$=0.4526752601\cdots,$$

$$\sum_{n=1}^{7}\frac{15600}{(36n-34)(36n-33)(36n-30)(36n-27)\cdots(36n-21)}$$
$$=0.2674920561\cdots.$$

即得 $A(1,18)=0.9638842503\cdots.$

直接计算 $A(1,18)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{18n-17}$，很难达到这样高的精确度．例如：

$$\sum_{n=1}^{370}(-1)^{n-1}\frac{1}{18n-17}=0.9638001223\cdots,$$

这里只准确到小数点后四位数。

仿此求得

$$A(1,8)$$
$$=\sum_{n=1}^{\infty}\frac{270270}{(16n-15)(16n-14)(16n-12)\cdots(16n-2)}$$
$$+\frac{719\pi}{1024\sqrt{2}}-\frac{207\pi}{512}+\frac{5945}{2048}\ln 2$$
$$-\frac{2951}{1024\sqrt{2}}\ln(\sqrt{2}+1).$$

而 $$\sum_{n=1}^{6}\frac{270270}{(16n-15)(16n-14)(16n-12)\cdots(16-2)}$$
$$=0.4189491976\cdots,$$

所以 $A(1,8)=0.9246517057\cdots;$

$A(1,9)$

$$=\sum_{n=1}^{\infty}\frac{1760}{(18n-17)(18n-16)(18n-15)(18n-12)\cdots(18n-6)}$$

$$+\sum_{n=1}^{\infty}\frac{15600}{(18n-16)(18n-15)(18n-12)(18n-9)\cdots(18n-3)}$$

$$+\frac{107\pi}{1458\sqrt{3}}+\frac{689}{729}\ln 2-\frac{85}{162}\ln 3,$$

而 $$\sum_{n=1}^{10}\frac{1760}{(18n-17)(18n-16)(18n-15)(18n-12)\cdots(18n-6)}$$

$$=0.4526866482\cdots,$$

$$\sum_{n=1}^{13}\frac{15600}{(18n-16)(18n-15)(18n-12)(18n-9)\cdots(18n-3)}$$

$$=0.2675502583\cdots,$$

所以 $A(1,9)=0.9320304221\cdots$;

$A(1,12)$

$$=\sum_{n=1}^{\infty}\frac{420}{(24n-23)(24n-22)\cdots(24n-20)(24n-18)(24n-16)}$$

$$+\frac{187\pi}{192\sqrt{3}}-\frac{7\pi}{12\sqrt{2}}-\frac{3\pi}{64}-\frac{17}{16}\ln 2$$

$$+\frac{63}{64}\ln 3+\frac{9\sqrt{3}}{32}\ln(2+\sqrt{3})$$

$$-\frac{7}{6\sqrt{2}}\ln(\sqrt{2}+1),$$

而

$$\sum_{n=1}^{6}\frac{420}{(24n-23)(24n-22)\cdots(24n-20)(24n-18)(24n-16)}$$

$$=0.3645842477\cdots,$$

所以 $A(1,12)=0.9474696544\cdots$;

$A(1,15)$

$$= \sum_{n=1}^{\infty} \frac{19152}{(30n-29)(30n-28)(30n-25)(30n-20)\cdots(30n-10)}$$

$$+ \sum_{n=1}^{\infty} \frac{182160}{(30n-28)(30n-25)(30n-20)(30n-15)\cdots(30n-5)}$$

$$+ \frac{56\pi}{1875\sqrt{3}} + \frac{829}{1875}\ln 2 - \frac{162}{625}\ln 3,$$

而 $\sum_{n=1}^{9} \frac{19152}{(30n-29)(30n-28)(30n-25)(30n-20)\cdots(30n-10)}$

$= 0.6384063414\cdots,$

$\sum_{n=1}^{12} \frac{182160}{(30n-28)(30n-25)(30n-20)(30n-15)\cdots(30n-5)}$

$= 0.2429143299\cdots,$

所以 $A(1,15) = 0.9571959804\cdots;$

$A(1,16)$

$$= \sum_{n=1}^{\infty} \frac{6930}{(32n-31)(32n-30)(32n-28)(32n-24)\cdots(32n-16)}$$

$$+ \frac{17}{32} \sum_{n=1}^{\infty} \frac{1890}{(16n-15)(16n-14)(16n-12)(16n-10)\cdots(16n-6)}$$

$$+ \frac{3801\pi}{16384\sqrt{2}} - \frac{1109\pi}{8192} + \frac{2429}{8192}\ln 2$$

$$- \frac{2121}{8192\sqrt{2}}\ln(\sqrt{2}+1),$$

而

$\sum_{n=1}^{7} \frac{6930}{(32n-31)(32n-30)(32n-28)(32n-24)\cdots(32n-16)}$

$= 0.5639669362\cdots,$

$$\sum_{n=1}^{10}\frac{1890}{(16n-15)(16n-14)(16n-12)(16n-10)\cdots(16n-6)}$$

$$=0.4922108387\cdots,$$

所以 $A(1,16)=0.9596845382\cdots$;

$A(1,24)$

$$=\sum_{n=1}^{\infty}\frac{43010}{(48n-47)(48n-46)(48n-42)(48n-36)\cdots(48n-24)}$$

$$+\sum_{n=1}^{\infty}\frac{415800}{(48n-46)(48n-42)(48n-36)(48n-30)\cdots(48n-18)}$$

$$+\frac{15169\pi}{124416\sqrt{2}}-\frac{2245\pi}{31104}+\frac{9295}{62208}\ln 2$$

$$-\frac{8921}{62208\sqrt{2}}\ln(\sqrt{2}+1),$$

而

$$\sum_{n=1}^{4}\frac{43010}{(48n-47)(48n-46)(48n-42)(48n-36)\cdots(48n-24)}$$

$$=0.6913913490\cdots,$$

$$\sum_{n=1}^{8}\frac{415800}{(48n-46)(48n-42)(48n-36)(48n-30)\cdots(48n-18)}$$

$$=0.2228081072\cdots,$$

所以 $A(1,24)=0.9724842827\cdots$。

6. $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\boldsymbol{A}}{\boldsymbol{k(An-A)+a}}-\frac{\boldsymbol{B}}{\boldsymbol{k(Bn-B)+b}}\right]$

定理14 设 A、B、b 都是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(An-A)+a}-\frac{B}{k(Bn-B)+b}\right]$$

$$=AC(a,Ak)-BC(b,Bk)+\frac{1}{k}\ln\frac{Ab}{Ba}.$$

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{A}{k(Ai-A)+a}-\frac{B}{k(Bi-B)+b}\right]$$

$$=AC(a,Ak)-BC(b,Bk)+\frac{1}{k}\ln\frac{Ank-Ak+a}{Bnk-Bk+b}$$

$$+\frac{1}{k}\ln\frac{b}{a}+A\varepsilon_n(a,Ak)-B\varepsilon_n(b,Bk).$$

令 $n\to\infty$并取极限，定理即得证.

定理9是定理14的特殊情形，于定理14令$A=1$，$B=1$，即得定理9.

例1 求和：

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理14 令$A=2$，$B=3$，$a=1$，$b=2$，$k=1$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)}=\sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{3}{3n-1}\right)$$

$$=2C(1,2)-3C(2,3)+\ln\frac{2^2}{3}.$$

根据定理6，有

$$2C(1,2)-3C(2,3)=\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\ln 3.$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)}=\frac{\pi}{2\sqrt{3}}+2\ln 2-\frac{3}{2}\ln 3.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-1)}=\sum_{n=1}^{\infty}\left(\frac{3}{3n-1}-\frac{4}{4n-1}\right)$$

$$=-\frac{\pi}{2\sqrt{3}}+\frac{\pi}{2}+\frac{3}{2}\ln 3-3\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-1)}=\sum_{n=1}^{\infty}\left(\frac{2}{4n-1}-\frac{3}{6n-1}\right)$$

$$=\frac{\pi\sqrt{3}}{4}-\frac{\pi}{4}+\frac{1}{2}\ln 2-\frac{3}{4}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-1)}=\sum_{n=1}^{\infty}\left(\frac{3}{6n-1}-\frac{4}{8n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{4}+\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\frac{3}{4}\ln 3-\ln 2$$

$$-\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-1)(12n-1)}=\sum_{n=1}^{\infty}\left(\frac{2}{8n-1}-\frac{3}{12n-1}\right)$$

$$=\frac{\pi\sqrt{3}}{8}-\frac{\pi}{4\sqrt{2}}+\frac{\pi}{8}+\frac{1}{4}\ln 2-\frac{3}{8}\ln 3$$

$$+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)}-\frac{2}{(3n-1)(4n-1)}\right]$$

$$=\frac{\pi\sqrt{3}}{2}-\pi+8\ln 2-\frac{9}{2}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-1)(6n-1)}$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{(3n-1)(4n-1)}-\frac{2}{(4n-1)(6n-1)}\right]$$

$$=-\frac{2\pi}{\sqrt{3}}+\pi+3\ln 3-4\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-1)(8n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-1)}-\frac{2}{(6n-1)(8n-1)}\right]$$

$$=\frac{3\pi\sqrt{3}}{4}-\frac{\pi}{\sqrt{2}}-\frac{3\pi}{4}+\frac{5}{2}\ln 2-\frac{9}{4}\ln 3$$

$$+\sqrt{2}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-1)(12n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-1)}-\frac{2}{(8n-1)(12n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{2}+\frac{\pi}{\sqrt{2}}+\frac{3}{2}\ln 3-\frac{3}{2}\ln 2$$

$$+\frac{\sqrt{3}}{2}\ln(2+\sqrt{3})-\sqrt{2}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)(4n-1)}\right.$$

$$\left.-\frac{3}{(3n-1)(4n-1)(6n-1)}\right]$$

$$= \frac{5\pi\sqrt{3}}{4} - 2\pi + 10\ln 2 - \frac{27}{4}\ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(3n-1)(4n-1)(6n-1)}\right.$$

$$\left. - \frac{8}{(4n-1)(6n-1)(8n-1)}\right]$$

$$= -\frac{8\pi\sqrt{3}}{5} + \frac{4\pi\sqrt{2}}{5} + \frac{9\pi}{5} - \frac{32}{5}\ln 2 + \frac{27}{5}\ln 3$$

$$- \frac{8\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-1)(8n-1)}\right.$$

$$\left. - \frac{3}{(6n-1)(8n-1)(12n-1)}\right]$$

$$= \frac{9\pi\sqrt{3}}{8} - \pi\sqrt{2} - \frac{3\pi}{8} + \frac{7}{2}\ln 2 - \frac{27}{8}\ln 3$$

$$+ 2\sqrt{2}\ln(\sqrt{2}+1) - \frac{3\sqrt{3}}{4}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$= \frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)}\right.$$

$$\left. - \frac{4}{(3n-1)(4n-1)(6n-1)(8n-1)}\right]$$

$$=\frac{51\pi\sqrt{3}}{20}-\frac{16\pi\sqrt{2}}{15}-\frac{46\pi}{15}+\frac{178}{15}\ln 2$$

$$-\frac{189}{20}\ln 3+\frac{32\sqrt{3}}{15}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(3n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{4}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{61\pi}{10\sqrt{3}}+\frac{8\pi\sqrt{2}}{5}+\frac{11\pi}{10}-\frac{34}{5}\ln 2+\frac{63}{10}\ln 3$$

$$-\frac{16\sqrt{2}}{5}\ln(\sqrt{2}+1)+\sqrt{3}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{59\pi\sqrt{3}}{20}-\frac{32\pi\sqrt{2}}{15}-\frac{29\pi}{15}+\frac{158}{15}\ln 2-\frac{189}{20}\ln 3$$

$$+\frac{64\sqrt{2}}{15}\ln(\sqrt{2}+1)-\frac{6\sqrt{3}}{5}\ln(2+\sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(4n-1)(6n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(4n-1)}-\frac{3}{(4n-1)(6n-1)}\right]$$

$$= -\frac{3\pi\sqrt{3}}{8} + \frac{\pi}{2} - \ln 2 + \frac{9}{8}\ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(6n-1)}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)} - \frac{3}{(3n-1)(6n-1)}\right]$$

$$= -\frac{\pi}{4\sqrt{3}} + 2\ln 2 - \frac{3}{4}\ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(6n-1)(8n-1)}$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(3n-1)(6n-1)} - \frac{8}{(6n-1)(8n-1)}\right]$$

$$= \frac{7\pi}{5\sqrt{3}} - \frac{2\pi\sqrt{2}}{5} - \frac{2\pi}{5} + \frac{6}{5}\ln 2 - \frac{6}{5}\ln 3$$

$$+ \frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(8n-1)}$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(3n-1)(4n-1)} - \frac{8}{(4n-1)(8n-1)}\right]$$

$$= -\frac{\pi\sqrt{3}}{10} - \frac{\pi\sqrt{2}}{5} + \frac{3\pi}{10} - \frac{7}{5}\ln 2 + \frac{9}{10}\ln 3$$

$$+ \frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(8n-1)(12n-1)}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(8n-1)} - \frac{3}{(8n-1)(12n-1)}\right]$$

$$= -\frac{3\pi\sqrt{3}}{16} + \frac{\pi}{2\sqrt{2}} - \frac{3\pi}{16} - \frac{1}{2}\ln 2 + \frac{9}{16}\ln 3$$

$$-\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)+\frac{3\sqrt{3}}{8}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-1)}-\frac{3}{(6n-1)(12n-1)}\right]$$

$$=\frac{\pi\sqrt{3}}{8}-\frac{3\pi}{8}+\frac{1}{2}\ln 2-\frac{3}{8}\ln 3$$

$$+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{4}{(4n-1)(6n-1)(8n-1)}\right]$$

$$=-\frac{9\pi\sqrt{3}}{8}+\frac{2\pi\sqrt{2}}{3}+\frac{7\pi}{6}-\frac{11}{3}\ln 2$$

$$+\frac{27}{8}\ln 3-\frac{4\sqrt{2}}{3}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)(6n-1)}\right.$$

$$\left.-\frac{4}{(3n-1)(6n-1)(8n-1)}\right]$$

$$=-\frac{13\pi\sqrt{3}}{20}+\frac{8\pi\sqrt{2}}{15}+\frac{8\pi}{15}-\frac{14}{15}\ln 2+\frac{27}{20}\ln 3$$

$$-\frac{16\sqrt{2}}{15}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(3n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{4}{(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{37\pi}{15\sqrt{3}}-\frac{4\pi\sqrt{2}}{5}-\frac{2\pi}{15}+\frac{12}{5}\ln 2-\frac{12}{5}\ln 3$$

$$+\frac{8\sqrt{2}}{5}\ln(\sqrt{2}+1)-\frac{2}{\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)(4n-1)}\right.$$

$$\left.-\frac{4}{(3n-1)(4n-1)(8n-1)}\right]$$

$$=\frac{3\pi\sqrt{3}}{10}+\frac{4\pi\sqrt{2}}{15}-\frac{11\pi}{15}+\frac{68}{15}\ln 2-\frac{27}{10}\ln 3$$

$$-\frac{8\sqrt{2}}{15}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(3n-1)(4n-1)(8n-1)}\right.$$

$$\left.-\frac{4}{(4n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{13\pi}{20\sqrt{3}}-\frac{2\pi\sqrt{2}}{5}+\frac{7\pi}{20}+\frac{1}{5}\ln 2-\frac{9}{20}\ln 3$$

$$+\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{2}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(3n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{4}{(4n-1)(6n-1)(12n-1)}\right]$$

$$=-\frac{7\pi}{6\sqrt{3}}+\frac{5\pi}{6}+\frac{3}{2}\ln 3-2\ln 2$$

$$-\frac{1}{\sqrt{3}}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{63\pi\sqrt{3}}{40}+\frac{4\pi\sqrt{2}}{3}+\frac{41\pi}{60}-\frac{74}{15}\ln 2+\frac{189}{40}\ln 3$$

$$-\frac{8\sqrt{2}}{3}\ln(\sqrt{2}+1)+\frac{9\sqrt{3}}{10}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)(3n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6}{(3n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$= -\frac{67\pi}{20\sqrt{3}} + \frac{16\pi\sqrt{2}}{15} + \frac{4\pi}{15} - \frac{46}{15}\ln 2 + \frac{63}{20}\ln 3$$

$$- \frac{32\sqrt{2}}{15}\ln(\sqrt{2}+1) + \frac{4\sqrt{3}}{5}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\Big[\frac{1}{(2n-1)(3n-1)(4n-1)(8n-1)}$$

$$- \frac{6}{(3n-1)(4n-1)(8n-1)(12n-1)}\Big]$$

$$= -\frac{\pi\sqrt{3}}{5} + \frac{8\pi\sqrt{2}}{15} - \frac{17\pi}{30} + \frac{2}{3}\ln 2$$

$$- \frac{16\sqrt{2}}{15}\ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{5}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$= \frac{1}{5}\sum_{n=1}^{\infty}\Big[\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$- \frac{6}{(3n-1)(4n-1)(6n-1)(12n-1)}\Big]$$

$$= \frac{43\pi}{20\sqrt{3}} - \frac{7\pi}{5} + \frac{22}{5}\ln 2 - \frac{63}{20}\ln 3$$

$$+ \frac{2\sqrt{3}}{5}\ln(2+\sqrt{3}).$$

例2 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-1)(4n-1)(6n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)}.$$

解 于定理14令 $A=4$，$B=6$，$a=1$，$b=2$，$k=1$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{4n-3}-\frac{3}{3n-2}\right)$$

$$=2C(1,4)-\frac{3}{2}C(1,3)+\frac{1}{2}\ln\frac{2^2}{3}.$$

根据定理6，有

$$2C(1,4)-\frac{3}{2}C(1,3)=-\frac{\pi}{4\sqrt{3}}+\frac{\pi}{4}+\frac{1}{4}\ln\frac{2^2}{3}.$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)}=-\frac{\pi}{4\sqrt{3}}+\frac{\pi}{4}+\frac{3}{2}\ln 2-\frac{3}{4}\ln 3.$$

仿此推得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(8n-5)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{3n-2}-\frac{8}{8n-5}\right)$$

$$=\frac{\pi}{4\sqrt{3}}-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\frac{3}{4}\ln 3-2\ln 2+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(12n-7)}=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{8}{8n-5}-\frac{12}{12n-7}\right)$$

$$=\frac{\pi\sqrt{3}}{8}+\frac{\pi}{4\sqrt{2}}-\frac{3\pi}{8}+\frac{1}{4}\ln 2-\frac{3}{8}\ln 3-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)(8n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-4)}-\frac{2}{(6n-4)(8n-5)}\right]$$

$$=-\frac{\pi\sqrt{3}}{4}+\frac{\pi}{\sqrt{2}}-\frac{\pi}{4}+\frac{11}{2}\ln 2-\frac{9}{4}\ln 3$$

$$-\sqrt{2}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(8n-5)(12n-7)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-4)(8n-5)}-\frac{2}{(8n-5)(12n-7)}\right]$$

$$=-\frac{\pi}{2\sqrt{3}}-\frac{\pi}{\sqrt{2}}+\pi+\frac{3}{2}\ln 3-\frac{5}{2}\ln 2$$

$$+\sqrt{2}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{2}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{3}{(6n-4)(8n-5)(12n-7)}\right]$$

$$=\frac{\pi\sqrt{3}}{8}+\pi\sqrt{2}-\frac{13\pi}{8}+\frac{13}{2}\ln 2-\frac{27}{8}\ln 3$$

$$-2\sqrt{2}\ln(\sqrt{2}+1)+\frac{3\sqrt{3}}{4}\ln(2+\sqrt{3}).$$

于是 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-1)(4n-1)}$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-4)(8n-5)}\right.$$

$$-\frac{1}{(4n-1)(6n-1)(8n-1)}\Bigg]$$

$$= -\pi\sqrt{3}+\pi\sqrt{2}+\frac{\pi}{2}+3\ln 2-2\sqrt{2}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-1)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(6n-4)(8n-5)(12n-7)}$$

$$-\frac{1}{(6n-1)(8n-1)(12n-1)}\Bigg]$$

$$=\frac{\pi}{\sqrt{3}}-\pi\sqrt{2}+\pi-\ln 2+2\sqrt{2}\ln(\sqrt{2}+1)$$

$$-\sqrt{3}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$-\frac{1}{(4n-1)(6n-1)(8n-1)(12n-1)}\Bigg]$$

$$=-\pi\sqrt{3}+2\pi\sqrt{2}-\frac{5\pi}{4}+3\ln 2$$

$$-4\sqrt{2}\ln(\sqrt{2}+1)+\frac{3\sqrt{3}}{2}\ln(2+\sqrt{3}).$$

例3 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(4n-1)(6n-1)}.$$

解 于定理14令$A=6$，$B=8$，$a=1$，$b=3, k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(8n-5)}=\frac{1}{10}\sum_{n=1}^{\infty}\left(\frac{6}{6n-5}-\frac{8}{8n-5}\right)$$

$$=\frac{\pi\sqrt{3}}{20}-\frac{\pi}{10\sqrt{2}}+\frac{\pi}{20}+\frac{3}{20}\ln 3-\frac{1}{5}\ln 2$$

$$+\frac{1}{5\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此推得

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(12n-7)}=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{8}{8n-5}-\frac{12}{12n-7}\right)$$

$$=\frac{\pi\sqrt{3}}{8}+\frac{\pi}{4\sqrt{2}}-\frac{3\pi}{8}+\frac{1}{4}\ln 2-\frac{3}{8}\ln 3$$

$$-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(8n-5)(12n-7)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(8n-5)}-\frac{2}{(8n-5)(12n-7)}\right]$$

$$=-\frac{\pi}{5\sqrt{3}}-\frac{\pi}{5\sqrt{2}}+\frac{4\pi}{15}+\frac{3}{10}\ln 3-\frac{7}{30}\ln 2$$

$$+\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1)-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}).$$

于定理14令 $A=6$, $B=8$, $a=4$, $b=7$, $k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(8n-1)}=\frac{1}{10}\sum_{n=1}^{\infty}\left(\frac{3}{3n-1}-\frac{8}{8n-1}\right)$$

$$=-\frac{\pi}{20\sqrt{3}}+\frac{\pi}{10\sqrt{2}}+\frac{\pi}{20}+\frac{3}{20}\ln 3-\frac{2}{5}\ln 2$$

$$-\frac{1}{5\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此推得

$$\sum_{n=1}^{\infty} \frac{1}{(8n-1)(12n-1)} = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{8}{8n-1} - \frac{12}{12n-1} \right)$$

$$= \frac{\pi\sqrt{3}}{8} - \frac{\pi}{4\sqrt{2}} + \frac{\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3$$

$$+ \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) - \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}).$$

于是 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-1)(12n-1)}$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \left[\frac{1}{(6n-2)(8n-1)} - \frac{2}{(8n-1)(12n-1)} \right]$$

$$= -\frac{4\pi}{15\sqrt{3}} + \frac{\pi}{5\sqrt{2}} - \frac{\pi}{15} + \frac{3}{10} \ln 3 - \frac{3}{10} \ln 2$$

$$- \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1) + \frac{1}{2\sqrt{3}} \ln(2+\sqrt{3}).$$

从而 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(4n-1)(6n-1)}$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-5)(8n-5)(12n-7)} - \frac{1}{(6n-2)(8n-1)(12n-1)} \right]$$

$$= \frac{\pi}{15\sqrt{3}} - \frac{\pi\sqrt{2}}{5} + \frac{\pi}{3} + \frac{1}{15} \ln 2 + \frac{2\sqrt{2}}{5} \ln(\sqrt{2}+1)$$

$$- \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}).$$

从以上的结果还可推得

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-5)(8n-5)(12n-7)} \right.$$

$$+\frac{1}{(6n-2)(8n-1)(12n-1)}\Bigg]$$

$$=-\frac{7\pi}{15\sqrt{3}}+\frac{\pi}{5}+\frac{3}{5}\ln 3-\frac{8}{15}\ln 2.$$

例4 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(4n-3)(4n-1)}.$$

解 于定理14令 $A=4$，$B=8$，$a=1$，$b=1$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(8n-7)}=-\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{4}{4n-3}-\frac{8}{8n-7}\right)$$

$$=\frac{\pi}{4\sqrt{2}}+\frac{1}{4}\ln 2+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1).$$

仿此推得

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-5}\right)$$

$$=\frac{\pi}{16}+\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(8n-3)}=-\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{4}{4n-1}-\frac{8}{8n-3}\right)$$

$$=-\frac{\pi}{4\sqrt{2}}+\frac{\pi}{4}+\frac{1}{4}\ln 2-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-1)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{8n-3}-\frac{1}{8n-1}\right)$$

$$=\frac{\pi}{16}-\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1).$$

欧立常数？

于是 $$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(8n-7)(8n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(8n-7)}-\frac{2}{(8n-7)(8n-5)}\right]$$

$$=\frac{\pi}{4\sqrt{2}}-\frac{\pi}{8}+\frac{1}{4}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(8n-3)(8n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(8n-3)}-\frac{2}{(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi}{4\sqrt{2}}+\frac{\pi}{8}+\frac{1}{4}\ln 2.$$

从而 $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(8n-7)(8n-5)}-\frac{1}{(4n-1)(8n-3)(8n-1)}\right]$$

$$=\frac{\pi}{2\sqrt{2}}-\frac{\pi}{4}.$$

例5 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-1)(6n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)(6n-1)},$$

解 于定理14令 $A=4$，$B=6$，$a=1$，$b=1$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)}=-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{4n-3}-\frac{6}{6n-5}\right)$$

$$=\frac{\pi\sqrt{3}}{4}-\frac{\pi}{4}+\frac{3}{4}\ln 3-\frac{1}{2}\ln 2.$$

仿此推得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)}=\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}-\frac{1}{6n-4}\right)$$

$$=\frac{\pi}{6\sqrt{3}}+\frac{1}{3}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(8n-5)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{3n-2}-\frac{8}{8n-5}\right)$$

$$=\frac{\pi}{4\sqrt{3}}-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\frac{3}{4}\ln 3-2\ln 2$$

$$+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(12n-7)}=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{8}{8n-5}-\frac{12}{12n-7}\right)$$

$$=\frac{\pi\sqrt{3}}{8}+\frac{\pi}{4\sqrt{2}}-\frac{3\pi}{8}+\frac{1}{4}\ln 2-\frac{3}{8}\ln 3$$

$$-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)}=-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{4n-1}-\frac{6}{6n-2}\right)$$

$$=-\frac{\pi}{4\sqrt{3}}+\frac{\pi}{4}+\frac{3}{4}\ln 3-\frac{3}{2}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)}=\sum_{n=1}^{\infty}\left(\frac{1}{6n-2}-\frac{1}{6n-1}\right)$$

$$=\frac{\pi}{6\sqrt{3}}-\frac{1}{3}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-1)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{6}{6n-1}-\frac{8}{8n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{4}+\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\frac{3}{4}\ln 3-\ln 2$$

$$-\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-1)(12n-1)}=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{8}{8n-1}-\frac{12}{12n-1}\right)$$

$$=\frac{\pi\sqrt{3}}{8}-\frac{\pi}{4\sqrt{2}}+\frac{\pi}{8}+\frac{1}{4}\ln 2-\frac{3}{8}\ln 3$$

$$+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2}{(4n-3)(6n-5)}-\frac{3}{(6n-5)(6n-4)}\right]$$

$$=\frac{\pi}{\sqrt{3}}-\frac{\pi}{2}+\frac{3}{2}\ln 3-2\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-5)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(6n-5)(6n-4)}-\frac{4}{(6n-4)(8n-5)}\right]$$

$$=-\frac{\pi}{10\sqrt{3}}+\frac{\pi\sqrt{2}}{5}-\frac{\pi}{5}+\frac{9}{5}\ln 2-\frac{3}{5}\ln 3$$

$$-\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(8n-5)(12n-7)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-4)(8n-5)}-\frac{2}{(8n-5)(12n-7)}\right]$$

$$=-\frac{\pi}{2\sqrt{3}}-\frac{\pi}{\sqrt{2}}+\pi+\frac{3}{2}\ln 3-\frac{5}{2}\ln 2$$

$$+\sqrt{2}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{2}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2}{(4n-1)(6n-2)}-\frac{3}{(6n-2)(6n-1)}\right]$$

$$=-\frac{\pi}{\sqrt{3}}+\frac{\pi}{2}+\frac{3}{2}\ln 3-2\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)(8n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(6n-2)(6n-1)}-\frac{4}{(6n-1)(8n-1)}\right]$$

$$=\frac{7\pi}{10\sqrt{3}}-\frac{\pi\sqrt{2}}{5}-\frac{\pi}{5}+\frac{3}{5}\ln 2-\frac{3}{5}\ln 3$$

$$+\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-1)(12n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-1)}-\frac{2}{(8n-1)(12n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{2}+\frac{\pi}{\sqrt{2}}+\frac{3}{2}\ln 3-\frac{3}{2}\ln 2$$

$$-\sqrt{2}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{2}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)}\right.$$

$$\left.-\frac{2}{(6n-5)(6n-4)(8n-5)}\right]$$

$$=\frac{2\pi\sqrt{3}}{5}-\frac{2\pi\sqrt{2}}{5}-\frac{\pi}{10}+\frac{27}{10}\ln 3-\frac{28}{5}\ln 2$$

$$+\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-5)(12n-7)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{2}{(6n-4)(8n-5)(12n-7)}\right]$$

$$=\frac{\pi\sqrt{3}}{10}+\frac{2\pi\sqrt{2}}{5}-\frac{11\pi}{15}+\frac{34}{15}\ln 2-\frac{6}{5}\ln 3$$

$$-\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1)+\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-2)(6n-1)}\right.$$

$$\left.-\frac{2}{(6n-2)(6n-1)(8n-1)}\right]$$

$$=-\frac{4\pi\sqrt{3}}{5}+\frac{2\pi\sqrt{2}}{5}+\frac{9\pi}{10}+\frac{27}{10}\ln 3-\frac{16}{5}\ln 2$$

$$-\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-2)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{2}{(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{37\pi}{30\sqrt{3}}-\frac{2\pi\sqrt{2}}{5}-\frac{\pi}{15}+\frac{6}{5}\ln 2-\frac{6}{5}\ln 3$$

$$+\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1)-\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)(12n-7)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{3}{(6n-5)(6n-4)(8n-5)(12n-7)}\right]$$

$$=\frac{\pi\sqrt{3}}{20}-\frac{4\pi\sqrt{2}}{5}+\frac{21\pi}{20}+\frac{63}{20}\ln 3-\frac{31}{5}\ln 2$$

$$+\frac{8\sqrt{2}}{5}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{2}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{3}{(6n-2)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{61\pi}{20\sqrt{3}}+\frac{4\pi\sqrt{2}}{5}+\frac{11\pi}{20}+\frac{63}{20}\ln 3-\frac{17}{5}\ln 2$$

$$-\frac{8\sqrt{2}}{5}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{2}\ln(2+\sqrt{3}).$$

于是

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)}\right.$$

$$\left.-\frac{1}{(4n-1)(6n-2)(6n-1)}\right]$$

$$=\frac{2\pi}{\sqrt{3}}-\pi,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{1}{(6n-2)(6n-1)(8n-1)}\right]$$

$$=-\frac{4\pi}{5\sqrt{3}}+\frac{2\pi\sqrt{2}}{5}+\frac{6}{5}\ln 2$$

$$-\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)}\right]$$

$$=\frac{6\pi\sqrt{3}}{5}-\frac{4\pi\sqrt{2}}{5}-\pi-\frac{12}{5}\ln 2$$

$$+\frac{8\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-5)(12n-7)}\right.$$

$$\left.-\frac{1}{(6n-2)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{14\pi}{15\sqrt{3}}+\frac{4\pi\sqrt{2}}{5}-\frac{2\pi}{3}+\frac{16}{15}\ln 2$$

$$-\frac{8\sqrt{2}}{5}\ln(\sqrt{2}+1)+\frac{2}{\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)(12n-7)}\right.$$

$$\left.-\frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{16\pi}{5\sqrt{3}}-\frac{8\pi\sqrt{2}}{5}+\frac{\pi}{2}-\frac{14}{5}\ln 2$$

$$+\frac{16\sqrt{2}}{5}\ln(\sqrt{2}+1)-\sqrt{3}\ln(2+\sqrt{3}).$$

从以上的结果还可推得

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(3n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)}\right.$$

$$\left.+\frac{1}{(4n-1)(6n-2)(6n-1)}\right]$$

$$= 3 \ln 3 - 4 \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-5)(6n-4)(8n-5)} + \frac{1}{(6n-2)(6n-1)(8n-1)} \right]$$

$$= \frac{\pi\sqrt{3}}{5} - \frac{2\pi}{5} + \frac{12}{5} \ln 2 - \frac{6}{5} \ln 3;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)} + \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)} \right]$$

$$= -\frac{2\pi\sqrt{3}}{5} + \frac{4\pi}{5} + \frac{27}{5} \ln 3 - \frac{44}{5} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-5)(6n-4)(8n-5)(12n-7)} + \frac{1}{(6n-2)(6n-1)(8n-1)(12n-1)} \right]$$

$$= \frac{23\pi}{15\sqrt{3}} - \frac{4\pi}{5} + \frac{52}{15} \ln 2 - \frac{12}{5} \ln 3;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)(12n-7)} \right.$$

$$+\frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)(12n-1)}\Bigg]$$

$$=-\frac{29\pi}{10\sqrt{3}}+\frac{8\pi}{5}+\frac{63}{10}\ln 3-\frac{48}{5}\ln 2.$$

仿此求得

(一) $$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(8n-7)}$$

$$=-\frac{3\pi\sqrt{3}}{4}+\frac{\pi}{\sqrt{2}}+\frac{3\pi}{4}+\frac{5}{2}\ln 2-\frac{9}{4}\ln 3$$

$$+\sqrt{2}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(8n-7)(8n-5)}$$

$$=-\frac{3\pi\sqrt{3}}{20}+\frac{3\pi}{10\sqrt{2}}+\frac{\pi}{10}+\frac{3}{5}\ln 2-\frac{9}{20}\ln 3$$

$$\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(12n-7)}$$

$$=-\frac{3\pi\sqrt{3}}{56}-\frac{3\pi}{28\sqrt{2}}+\frac{5\pi}{28}-\frac{3}{28}\ln 2$$

$$+\frac{9}{56}\ln 3+\frac{\sqrt{2}}{7}\ln(\sqrt{2}+1)$$

$$-\frac{3\sqrt{3}}{28}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(8n-3)}$$

$$=\frac{\pi\sqrt{3}}{4}-\frac{\pi}{\sqrt{2}}+\frac{\pi}{4}+\frac{11}{2}\ln 2-\frac{9}{4}\ln 3$$

$$-\sqrt{2}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(8n-3)(8n-1)}$$

$$=\frac{\pi\sqrt{3}}{20}-\frac{3\pi}{10\sqrt{2}}+\frac{\pi}{10}+\frac{6}{5}\ln 2-\frac{9}{20}\ln 3$$

$$-\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-1)(12n-1)}$$

$$=-\frac{3\pi\sqrt{3}}{56}+\frac{3\pi}{28\sqrt{2}}-\frac{\pi}{28}-\frac{3}{28}\ln 2+\frac{9}{56}\ln 3$$

$$-\frac{\sqrt{2}}{7}\ln(\sqrt{2}+1)+\frac{3\sqrt{3}}{28}\ln(2+\sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(4n-3)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(8n-7)}\right.$$

$$\left.-\frac{1}{(4n-1)(6n-2)(8n-3)}\right]$$

$$=-\pi\sqrt{3}+\pi\sqrt{2}+\frac{\pi}{2}-3\ln 2+2\sqrt{2}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(4n-3)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(8n-7)}\right.$$

$$\left.+\frac{1}{(4n-1)(6n-2)(8n-3)}\right]$$

$$=-\frac{\pi\sqrt{3}}{2}+\pi+8\ln 2-\frac{9}{2}\ln 3,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{1}{(6n-2)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{5}+\frac{3\pi}{5\sqrt{2}}-\frac{3}{5}\ln 2+\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(8n-7)(8n-5)}\right.$$

$$\left.+\frac{1}{(6n-2)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{10}+\frac{\pi}{5}+\frac{9}{5}\ln 2-\frac{9}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-3)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(12n-7)}\right.$$

$$\left.-\frac{1}{(8n-3)(8n-1)(12n-1)}\right]$$

$$=-\frac{3\pi}{14\sqrt{2}}+\frac{3\pi}{14}+\frac{2\sqrt{2}}{7}\ln(\sqrt{2}+1)$$

$$-\frac{3\sqrt{3}}{14}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(12n-7)}\right.$$

$$+\frac{1}{(8n-3)(8n-1)(12n-1)}\Bigg]$$

$$=-\frac{3\pi\sqrt{3}}{28}+\frac{\pi}{7}-\frac{3}{14}\ln 2+\frac{9}{28}\ln 3.$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-3)(6n-5)(8n-7)}$$

$$-\frac{2}{(6n-5)(8n-7)(8n-5)}\Bigg]$$

$$=-\frac{9\pi\sqrt{3}}{20}+\frac{\pi\sqrt{2}}{5}+\frac{11\pi}{20}+\frac{13}{10}\ln 2$$

$$-\frac{27}{20}\ln 3+\frac{3\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(8n-7)(8n-5)(12n-7)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(6n-5)(8n-7)(8n-5)}$$

$$-\frac{2}{(8n-7)(8n-5)(12n-7)}\Bigg]$$

$$=-\frac{\pi\sqrt{3}}{70}+\frac{3\pi\sqrt{2}}{35}-\frac{3\pi}{35}+\frac{18}{70}\ln 2-\frac{9}{35}\ln 3$$

$$-\frac{\sqrt{2}}{35}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{14}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-1)(6n-2)(8n-3)}$$

$$- \frac{2}{(6n-2)(8n-3)(8n-1)} \Big]$$

$$= \frac{3\pi\sqrt{3}}{20} - \frac{\pi\sqrt{2}}{5} + \frac{\pi}{20} + \frac{31}{10} \ln 2 - \frac{27}{20} \ln 3$$

$$- \frac{3\sqrt{2}}{5} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-3)(8n-1)(12n-1)}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \Big[\frac{1}{(6n-2)(8n-3)(8n-1)}$$

$$- \frac{2}{(8n-3)(8n-1)(12n-1)} \Big]$$

$$= \frac{11\pi}{70\sqrt{3}} - \frac{3\pi\sqrt{2}}{35} + \frac{2\pi}{35} + \frac{33}{70} \ln 2 - \frac{9}{35} \ln 3$$

$$+ \frac{\sqrt{2}}{35} \ln(\sqrt{2}+1) - \frac{\sqrt{2}}{14} \ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \Big[\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)}$$

$$- \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)} \Big]$$

$$= -\frac{3\pi\sqrt{3}}{5} + \frac{2\pi\sqrt{2}}{5} + \frac{\pi}{2} - \frac{9}{5} \ln 2$$

$$+ \frac{6\sqrt{2}}{5} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \Big[\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)}$$

$$+\frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)}\Bigg]$$

$$=-\frac{3\pi\sqrt{3}}{10}+\frac{3\pi}{5}+\frac{22}{5}\ln 2-\frac{27}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(4n-3)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(6n-5)(8n-7)(8n-5)(12n-7)}$$

$$-\frac{1}{(6n-2)(8n-3)(8n-1)(12n-1)}\Bigg]$$

$$=-\frac{\pi}{5\sqrt{3}}+\frac{6\pi\sqrt{2}}{35}-\frac{\pi}{7}-\frac{1}{5}\ln 2$$

$$\frac{2\sqrt{2}}{35}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{7}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(4n-3)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(6n-5)(8n-7)(8n-5)(12n-7)}$$

$$+\frac{1}{(6n-2)(8n-3)(8n-1)(12n-1)}\Bigg]$$

$$=\frac{4\pi}{35\sqrt{3}}-\frac{\pi}{35}+\frac{26}{35}\ln 2-\frac{18}{35}\ln 3.$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)(12n-7)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)}$$

$$-\frac{3}{(6n-5)(8n-7)(8n-5)(12n-7)}\Bigg]$$

$$= -\frac{57\pi\sqrt{3}}{280} - \frac{\pi\sqrt{2}}{35} + \frac{113\pi}{280} + \frac{17}{70}\ln 2 - \frac{81}{280}\ln 3$$
$$+ \frac{12\sqrt{2}}{35}\ln(\sqrt{2}+1) - \frac{3\sqrt{3}}{28}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-7)(8n-3)(8n-1)(12n-1)}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)}\right.$$
$$\left. - \frac{3}{(6n-2)(8n-3)(8n-1)(12n-1)}\right]$$
$$= -\frac{\pi\sqrt{3}}{280} + \frac{\pi\sqrt{2}}{35} - \frac{17\pi}{280} + \frac{59}{70}\ln 2 - \frac{81}{280}\ln 3$$
$$- \frac{12\sqrt{2}}{35}\ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{28}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)(6n-1)}$$
$$= \sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)(12n-7)}\right.$$
$$\left. - \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)(12n-1)}\right]$$
$$= -\frac{\pi\sqrt{3}}{5} - \frac{2\pi\sqrt{2}}{35} + \frac{13\pi}{28} - \frac{3}{5}\ln 2$$
$$+ \frac{24\sqrt{2}}{35}\ln(\sqrt{2}+1) - \frac{3\sqrt{3}}{14}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)(6n-1)}$$
$$= \sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)(12n-7)}\right.$$
$$\left. + \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)(12n-1)}\right]$$

$$=-\frac{29\pi\sqrt{3}}{140}+\frac{12\pi}{35}+\frac{38}{35}\ln 2-\frac{81}{140}\ln 3.$$

（二） $$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)}$$

$$=\frac{9\pi\sqrt{3}}{8}-\pi\sqrt{2}-\frac{3\pi}{8}+\frac{27}{8}\ln 3-\frac{7}{2}\ln 2$$

$$-2\sqrt{2}\ln(\sqrt{2}+1)+\frac{3\sqrt{3}}{4}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(8n-7)(12n-11)(12n-7)}$$

$$=\frac{25\pi}{56\sqrt{3}}-\frac{\pi\sqrt{2}}{7}-\frac{\pi}{42}+\frac{3}{7}\ln 3-\frac{19}{42}\ln 2$$

$$+\frac{11}{28\sqrt{3}}\ln(2+\sqrt{3})-\frac{2\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)}$$

$$=\frac{\pi\sqrt{3}}{8}+\pi\sqrt{2}-\frac{13\pi}{8}+\frac{27}{8}\ln 3-\frac{13}{2}\ln 2$$

$$+2\sqrt{2}\ln(\sqrt{2}+1)-\frac{3\sqrt{3}}{4}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(8n-3)(12n-5)(12n-1)}$$

$$=\frac{19\pi}{168\sqrt{3}}+\frac{\pi\sqrt{2}}{7}-\frac{11\pi}{42}+\frac{3}{7}\ln 3-\frac{11}{14}\ln 2$$

$$+\frac{2\sqrt{2}}{7}\ln(\sqrt{2}+1)-\frac{11}{28\sqrt{3}}\ln(2+\sqrt{3}).$$

由此求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)}\right.$$

$$\left.-\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)}\right]$$

$$= \pi\sqrt{3} - 2\pi\sqrt{2} + \frac{5\pi}{4} + 3\ln 2 - 4\sqrt{2}\ln(\sqrt{2}+1)$$

$$+\frac{3\sqrt{3}}{2}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)}\right.$$

$$\left.+\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)}\right]$$

$$= \frac{5\pi\sqrt{3}}{4} - 2\pi + \frac{27}{4}\ln 3 - 10\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(4n-3)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(8n-7)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{1}{(6n-2)(8n-3)(12n-5)(12n-1)}\right]$$

$$= \frac{\pi}{3\sqrt{3}} - \frac{2\pi\sqrt{2}}{7} + \frac{5\pi}{21} + \frac{1}{3}\ln 2$$

$$+\frac{11}{14\sqrt{3}}\ln(2+\sqrt{3}) - \frac{4\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(4n-3)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(8n-7)(12n-11)(12n-7)}\right.$$

$$+\frac{1}{(6n-2)(8n-3)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{47\pi}{84\sqrt{3}}-\frac{2\pi}{7}+\frac{6}{7}\ln 3-\frac{26}{21}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)(12n-7)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)}$$

$$-\frac{3}{(6n-5)(8n-7)(12n-11)(12n-7)}\Bigg]$$

$$=\frac{19\pi\sqrt{3}}{56}-\frac{2\pi\sqrt{2}}{7}-\frac{17\pi}{112}+\frac{117}{112}\ln 3-\frac{15}{14}\ln 2$$

$$+\frac{5\sqrt{3}}{28}\ln(2+\sqrt{3})-\frac{4\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)(12n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)}$$

$$-\frac{3}{(6n-2)(8n-3)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{\pi}{56\sqrt{3}}+\frac{2\pi\sqrt{2}}{7}-\frac{47\pi}{112}+\frac{117}{112}\ln 3-\frac{29}{14}\ln 2$$

$$+\frac{4\sqrt{2}}{7}\ln(\sqrt{2}+1)-\frac{5\sqrt{3}}{28}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)(12n-7)}$$

$$-\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{\pi}{\sqrt{3}}-\frac{4\pi\sqrt{2}}{7}+\frac{15\pi}{56}+\ln 2+\frac{5\sqrt{3}}{14}\ln(2+\sqrt{3})$$

$$-\frac{8\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)(12n-7)}$$

$$+\frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{29\pi}{28\sqrt{3}}-\frac{4\pi}{7}+\frac{117}{56}\ln 3-\frac{22}{7}\ln 2.$$

(三) $$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-7)}$$

$$=-\frac{7\pi}{10\sqrt{3}}+\frac{\pi\sqrt{2}}{5}+\frac{\pi}{5}+\frac{3}{5}\ln 2-\frac{3}{5}\ln 3$$

$$+\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(8n-7)(8n-5)}$$

$$=-\frac{\pi\sqrt{3}}{20}+\frac{3\pi}{10\sqrt{2}}-\frac{\pi}{10}+\frac{6}{5}\ln 2-\frac{9}{20}\ln 3$$

$$-\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(12n-11)}$$

$$= \frac{3\pi\sqrt{3}}{56} - \frac{3\pi}{28\sqrt{2}} + \frac{\pi}{28} + \frac{9}{56}\ln 3 - \frac{3}{28}\ln 2$$

$$+ \frac{3\sqrt{3}}{28}\ln(2+\sqrt{3}) - \frac{\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(12n-11)(12n-7)}$$

$$= -\frac{3\pi\sqrt{3}}{112} - \frac{\pi}{14\sqrt{2}} + \frac{3\pi}{28} + \frac{3}{28}\ln 3 - \frac{1}{14}\ln 2$$

$$-\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3}) + \frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)(8n-3)}$$

$$= \frac{\pi}{10\sqrt{3}} - \frac{\pi\sqrt{2}}{5} + \frac{\pi}{5} + \frac{9}{5}\ln 2 - \frac{3}{5}\ln 3$$

$$-\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-3)(8n-1)}$$

$$= \frac{3\pi\sqrt{3}}{20} - \frac{3\pi}{10\sqrt{2}} - \frac{\pi}{10} + \frac{3}{5}\ln 2 - \frac{9}{20}\ln 3$$

$$+\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-1)(12n-5)}$$

$$= \frac{3\pi\sqrt{3}}{56} + \frac{3\pi}{28\sqrt{2}} - \frac{5\pi}{28} + \frac{9}{56}\ln 3 - \frac{3}{28}\ln 2$$

$$-\frac{3\sqrt{3}}{28}\ln(2+\sqrt{3}) + \frac{\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-1)(12n-5)(12n-1)}$$

$$= -\frac{3\pi\sqrt{3}}{112}+\frac{\pi}{14\sqrt{2}}-\frac{\pi}{28}+\frac{3}{28}\ln 3-\frac{1}{14}\ln 2$$

$$+\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3})-\frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1)。$$

由此顺次推得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-3)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)}\right.$$

$$\left.-\frac{1}{(6n-2)(6n-1)(8n-3)}\right]$$

$$=-\frac{4\pi}{5\sqrt{3}}+\frac{2\pi\sqrt{2}}{5}-\frac{6}{5}\ln 2+\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)(4n-3)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)}\right.$$

$$\left.+\frac{1}{(6n-2)(6n-1)(8n-3)}\right]$$

$$=-\frac{\pi\sqrt{3}}{5}+\frac{2\pi}{5}+\frac{12}{5}\ln 2-\frac{6}{5}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{1}{(6n-1)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{5}+\frac{3\pi}{5\sqrt{2}}+\frac{3}{5}\ln 2-\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.+\frac{1}{(6n-1)(8n-3)(8n-1)}\right]$$

$$=\frac{\pi\sqrt{3}}{10}-\frac{\pi}{5}+\frac{9}{5}\ln 2-\frac{9}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{1}{(8n-3)(8n-1)(12n-5)}\right]$$

$$=-\frac{3\pi}{14\sqrt{2}}+\frac{3\pi}{14}+\frac{3\sqrt{3}}{14}\ln(2+\sqrt{3})$$

$$-\frac{2\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.+\frac{1}{(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{3\pi\sqrt{3}}{28}-\frac{\pi}{7}+\frac{9}{28}\ln 3-\frac{3}{14}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-5)(12n-11)(12n-7)}\right.$$

$$-\frac{1}{(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=-\frac{\pi}{7\sqrt{2}}+\frac{\pi}{7}-\frac{3\sqrt{3}}{28}\ln(2+\sqrt{3})$$

$$+\frac{\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{1}{(8n-5)(12n-11)(12n-7)}$$

$$+\frac{1}{(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=-\frac{3\pi\sqrt{3}}{56}+\frac{\pi}{14}+\frac{3}{14}\ln 3-\frac{1}{7}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\Bigg[\frac{3}{(6n-5)(6n-4)(8n-7)}$$

$$-\frac{4}{(6n-4)(8n-7)(8n-5)}\Bigg]$$

$$=-\frac{\pi\sqrt{3}}{10}+\frac{\pi}{5}-\frac{3}{5}\ln 2+\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$=-\frac{1}{3}\sum_{n=1}^{\infty}\Bigg[\frac{1}{(6n-4)(8n-7)(8n-5)}$$

$$-\frac{2}{(8n-7)(8n-5)(12n-11)}\Bigg]$$

$$=\frac{11\pi}{70\sqrt{3}}-\frac{3\pi\sqrt{2}}{35}+\frac{2\pi}{35}+\frac{9}{35}\ln 3-\frac{33}{70}\ln 2$$

$$+\frac{\sqrt{3}}{14}\ln(2+\sqrt{3})-\frac{\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{2}{(8n-7)(8n-5)(12n-11)}-\frac{3}{(8n-5)(12n-11)(12n-7)}\right]$$

$$=\frac{3\pi\sqrt{3}}{112}-\frac{\pi}{28}+\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3})$$

$$-\frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(6n-2)(6n-1)(8n-3)}-\frac{4}{(6n-1)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{10}+\frac{\pi}{5}+\frac{3}{5}\ln 2-\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$=-\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-3)(8n-1)}-\frac{2}{(8n-3)(8n-1)(12n-5)}\right]$$

$$=-\frac{\pi\sqrt{3}}{70}+\frac{3\pi\sqrt{2}}{35}-\frac{3\pi}{35}-\frac{19}{70}\ln 2+\frac{9}{35}\ln 3$$

$$+\frac{\sqrt{2}}{35}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{14}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{2}{(8n-3)(8n-1)(12n-5)}\right.$$

$$\left.-\frac{3}{(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{3\pi\sqrt{3}}{112}-\frac{\pi}{28}+\frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1)$$

$$\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)}\right]$$

$$=-\frac{6}{5}\ln 2+\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.+\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{5}+\frac{2\pi}{5};$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{\pi}{5\sqrt{3}}-\frac{6\pi\sqrt{2}}{35}+\frac{\pi}{7}-\frac{1}{5}\ln 2+\frac{\sqrt{3}}{7}\ln(2+\sqrt{3})$$

$$-\frac{2\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.+\frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{4\pi}{35\sqrt{3}}-\frac{\pi}{35}+\frac{18}{35}\ln 3-\frac{26}{35}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{1}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{3\sqrt{3}}{28}\ln(2+\sqrt{3})-\frac{\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{1}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$= \frac{3\pi\sqrt{3}}{56} - \frac{\pi}{14}.$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$= -\sum_{n=1}^{\infty} \left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)} - \frac{2}{(6n-4)(8n-7)(8n-5)(12n-11)} \right]$$

$$= \frac{43\pi}{70\sqrt{3}} - \frac{6\pi\sqrt{2}}{35} - \frac{3\pi}{35} + \frac{18}{35}\ln 3 - \frac{12}{35}\ln 2 + \frac{\sqrt{3}}{7}\ln(2+\sqrt{3}) - \frac{16\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)} - \frac{2}{(8n-7)(8n-5)(12n-11)(12n-7)} \right]$$

$$= -\frac{\pi}{280\sqrt{3}} - \frac{3\pi\sqrt{2}}{35} + \frac{9\pi}{70} + \frac{9}{35}\ln 3 - \frac{33}{70}\ln 2 + \frac{4\sqrt{2}}{35}\ln(\sqrt{2}+1) - \frac{\sqrt{3}}{28}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$= -\sum_{n=1}^{\infty} \left[\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)} - \frac{2}{(6n-1)(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{\pi\sqrt{3}}{14} + \frac{6\pi\sqrt{2}}{35} - \frac{13\pi}{35} + \frac{18}{35}\ln 3 - \frac{8}{7}\ln 2$$

$$+ \frac{16\sqrt{2}}{35}\ln(\sqrt{2}+1) - \frac{\sqrt{3}}{7}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)}\right.$$

$$\left. - \frac{2}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$= -\frac{19\pi\sqrt{3}}{280} + \frac{3\pi\sqrt{2}}{35} - \frac{\pi}{70} + \frac{9}{35}\ln 3 - \frac{19}{70}\ln 2$$

$$- \frac{4\sqrt{2}}{35}\ln(\sqrt{2}+1) + \frac{\sqrt{3}}{28}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left. - \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$= \frac{2\pi}{5\sqrt{3}} - \frac{12\pi\sqrt{2}}{35} + \frac{2\pi}{7} + \frac{4}{5}\ln 2$$

$$+ \frac{2\sqrt{3}}{7}\ln(2+\sqrt{3}) - \frac{32\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left. + \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$= \frac{29\pi}{35\sqrt{3}} - \frac{16\pi}{35} + \frac{36}{35}\ln 3 - \frac{52}{35}\ln 2;$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right.$$

$$\left. - \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= \frac{\pi}{5\sqrt{3}} - \frac{6\pi\sqrt{2}}{35} + \frac{\pi}{7} - \frac{1}{5}\ln 2$$

$$+ \frac{8\sqrt{2}}{35}\ln(\sqrt{2}+1) - \frac{2\sqrt{3}}{28}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right.$$

$$\left. + \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= -\frac{29\pi}{140\sqrt{3}} + \frac{4\pi}{35} + \frac{18}{35}\ln 3 - \frac{26}{35}\ln 2.$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \frac{1}{3}\sum_{n=1}^{\infty} \left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right.$$

$$\left. - \frac{2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right]$$

$$= \frac{29\pi}{140\sqrt{3}} - \frac{4\pi}{35} + \frac{1}{5}\ln 2 + \frac{\sqrt{3}}{14}\ln(2+\sqrt{3}) \cdot$$

$$-\frac{8\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right.$$

$$\left.-\frac{2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{29\pi}{140\sqrt{3}}-\frac{4\pi}{35}-\frac{1}{5}\ln 2+\frac{8\sqrt{2}}{35}\ln(\sqrt{2}+1)$$

$$-\frac{\sqrt{3}}{14}\ln(2+\sqrt{3});$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{2}{5}\ln 2+\frac{\sqrt{3}}{7}\ln(2+\sqrt{3})-\frac{16\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{29\pi}{70\sqrt{3}}-\frac{8\pi}{35}.$$

(四) $$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-5)(6n-1)}$$

$$=-\frac{3\pi\sqrt{3}}{56}+\frac{\pi}{14}+\frac{3}{14}\ln 3-\frac{1}{7}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-1)(8n-7)}$$

$$=-\frac{9\pi\sqrt{3}}{136}+\frac{\pi\sqrt{2}}{17}+\frac{\pi}{17}+\frac{4}{17}\ln 2-\frac{3}{17}\ln 3$$

$$+\frac{2\sqrt{2}}{17}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-7)(8n-1)}$$

$$=\frac{3\pi\sqrt{3}}{68}-\frac{2\pi\sqrt{2}}{51}-\frac{2\pi}{51}+\frac{3}{17}\ln 2-\frac{9}{68}\ln 3$$

$$+\frac{3}{17\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-1)(12n-11)}$$

$$=\frac{3\pi\sqrt{3}}{152}-\frac{5\pi}{114\sqrt{2}}+\frac{\pi}{57}+\frac{9}{152}\ln 3-\frac{3}{76}\ln 2$$

$$+\frac{3\sqrt{3}}{76}\ln(2+\sqrt{3})-\frac{3}{38\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-1)(12n-11)(12n-1)}$$

$$=-\frac{9\pi\sqrt{3}}{760}+\frac{\pi}{38\sqrt{2}}-\frac{\pi}{95}+\frac{3}{76}\ln 3-\frac{1}{38}\ln 2$$

$$+\frac{\sqrt{3}}{38}\ln(2+\sqrt{3})-\frac{1}{19\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)}$$

$$=-\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-5)(6n-1)}-\frac{2}{(6n-5)(6n-1)(8n-7)}\right]$$

$$=-\frac{15\pi\sqrt{3}}{952}+\frac{2\pi\sqrt{2}}{85}+\frac{11\pi}{1190}+\frac{73}{595}\ln 2$$

$$-\frac{27}{238}\ln 3+\frac{4\sqrt{2}}{85}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)}$$

$$=\frac{1}{17}\sum_{n=1}^{\infty}\left[\frac{3}{(6n-5)(6n-1)(8n-7)}-\frac{4}{(6n-1)(8n-7)(8n-1)}\right]$$

$$=-\frac{51\pi\sqrt{3}}{2312}+\frac{\pi\sqrt{2}}{51}+\frac{\pi}{51},$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-7)(8n-1)(12n-11)}$$

$$=-\frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-7)(8n-1)}-\frac{2}{(8n-7)(8n-1)(12n-11)}\right]$$

$$=-\frac{\pi}{646\sqrt{3}}-\frac{\pi}{969\sqrt{2}}+\frac{8\pi}{969}-\frac{55}{1938}\ln 2+\frac{9}{323}\ln 3$$

$$+\frac{1}{38\sqrt{3}}\ln(2+\sqrt{3})-\frac{6\sqrt{2}}{323}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-1)(12n-11)(12n-1)}$$

$$=\frac{1}{19}\sum_{n=1}^{\infty}\left[\frac{2}{(8n-7)(8n-1)(12n-11)}\right.$$

$$\left.-\frac{3}{(8n-1)(12n-11)(12n-1)}\right]$$

$$=\frac{57\pi\sqrt{3}}{14440}-\frac{\pi}{114\sqrt{2}}+\frac{\pi}{285};$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)}$$

$$=-\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)}\right.$$

$$\left.-\frac{2}{(4n-1)(6n-5)(6n-1)(8n-7)}\right]$$

$$=-\frac{81\pi\sqrt{3}}{952}+\frac{4\pi\sqrt{2}}{85}+\frac{107\pi}{1190}+\frac{146}{595}\ln 2$$

$$-\frac{27}{119}\ln 3+\frac{8\sqrt{2}}{85}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)}\right.$$

$$\left.-\frac{2}{(6n-5)(6n-1)(8n-7)(8n-1)}\right]$$

$$=\frac{27\pi\sqrt{3}}{952}-\frac{4\pi\sqrt{2}}{255}-\frac{107\pi}{3570}+\frac{73}{595}\ln 2$$

$$-\frac{27}{238}\ln 3+\frac{4\sqrt{2}}{85}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)}$$

$$=-\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)}\right.$$

$$\left.-\frac{2}{(6n-1)(8n-7)(8n-1)(12n-11)}\right]$$

$$=\frac{163\pi}{2584\sqrt{3}}-\frac{20\pi\sqrt{2}}{969}-\frac{\pi}{323}-\frac{55}{969}\ln 2+\frac{18}{332}\ln 3$$

$$+\frac{1}{19\sqrt{3}}\ln(2+\sqrt{3})-\frac{12\sqrt{2}}{323}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-7)(8n-1)(12n-11)(12n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-7)(8n-1)(12n-11)}\right.$$

$$\left.-\frac{2}{(8n-7)(8n-1)(12n-11)(12n-1)}\right]$$

$$=-\frac{163\pi}{6460\sqrt{3}}+\frac{8\pi\sqrt{2}}{969}+\frac{2\pi}{1615}-\frac{55}{1938}\ln 2+\frac{9}{323}\ln 3$$

$$+\frac{1}{38\sqrt{3}}\ln(2+\sqrt{3})-\frac{6\sqrt{2}}{323}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)}\right.$$

$$\left.-\frac{2}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)}\right]$$

$$= -\frac{27\pi\sqrt{3}}{952} + \frac{4\pi\sqrt{2}}{255} + \frac{107\pi}{3570},$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)}$$

$$= -\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)} - \frac{3}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)}\right]$$

$$= \frac{157\pi\sqrt{3}}{36176} - \frac{28\pi\sqrt{2}}{4845} + \frac{1403\pi}{542640} - \frac{414}{11305}\ln 2 + \frac{1269}{36176}\ln 3 + \frac{\sqrt{3}}{152}\ln(2+\sqrt{3}) - \frac{32\sqrt{2}}{1615}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)(12n-1)}$$

$$= \frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)} - \frac{2}{(6n-1)(8n-7)(8n-1)(12n-11)(12n-1)}\right]$$

$$= \frac{163\pi}{12920\sqrt{3}} - \frac{4\pi\sqrt{2}}{969} - \frac{\pi}{1615}.$$

（五） $$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-5)}$$

$$= -\frac{3\pi\sqrt{3}}{748} - \frac{3\pi}{374\sqrt{2}} + \frac{7\pi}{374} - \frac{3}{187}\ln 2 + \frac{9}{748}\ln 3 + \frac{7\sqrt{2}}{187}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(12n-11)}$$

$$=\frac{3\pi\sqrt{3}}{56}-\frac{3\pi}{28\sqrt{2}}+\frac{\pi}{28}+\frac{9}{56}\ln 3-\frac{3}{28}\ln 2$$

$$+\frac{3\sqrt{3}}{28}\ln(2+\sqrt{3})-\frac{\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-5)(12n-11)(12n-7)}$$

$$=-\frac{3\pi\sqrt{3}}{112}-\frac{\pi}{14\sqrt{2}}+\frac{3\pi}{28}+\frac{3}{28}\ln 3-\frac{1}{14}\ln 2$$

$$-\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3})+\frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次推得

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3}{(6n-5)(6n-1)(8n-7)}\right.$$

$$\left.-\frac{4}{(6n-1)(8n-7)(8n-5)}\right]$$

$$=-\frac{273\pi\sqrt{3}}{7480}+\frac{36\pi\sqrt{2}}{935}+\frac{19\pi}{935}+\frac{144}{935}\ln 2$$

$$-\frac{108}{935}\ln 3+\frac{38\sqrt{2}}{935}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-7)(8n-5)(12n-11)}$$

$$=-\frac{1}{9}\sum_{n=1}^{\infty}\left[\frac{1}{(6n-1)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{2}{(8n-7)(8n-5)(12n-11)}\right]$$

$$= \frac{97\pi\sqrt{3}}{7854} - \frac{15\pi\sqrt{2}}{1309} + \frac{23\pi}{3927} + \frac{45}{1309}\ln 3 - \frac{173}{7854}\ln 2$$

$$+ \frac{1}{14\sqrt{3}}\ln(2+\sqrt{3}) - \frac{47\sqrt{2}}{1309}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{2}{(8n-7)(8n-5)(12n-11)}\right.$$

$$\left. - \frac{3}{(8n-5)(12n-11)(12n-7)}\right]$$

$$= \frac{3\pi\sqrt{3}}{112} - \frac{\pi}{28} + \frac{3\sqrt{3}}{56}\ln(2+\sqrt{3})$$

$$- \frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)}$$

$$= -\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)}\right.$$

$$\left. - \frac{2}{(6n-5)(6n-1)(8n-7)(8n-5)}\right]$$

$$= -\frac{999\pi\sqrt{3}}{52360} + \frac{10\pi\sqrt{2}}{561} + \frac{137\pi}{13090} + \frac{1213}{19635}\ln 2$$

$$- \frac{513}{13090}\ln 3 + \frac{32\sqrt{2}}{2805}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)}$$

$$= -\sum_{n=1}^{\infty}\left[\frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)}\right.$$

$$-\frac{2}{(6n-1)(8n-7)(8n-5)(12n-11)}\Big]$$

$$=\frac{9613\pi\sqrt{3}}{157080}-\frac{402\pi\sqrt{2}}{6545}-\frac{169\pi}{19635}-\frac{3889}{19635}\ln 2$$

$$+\frac{1206}{6545}\ln 3-\frac{736\sqrt{2}}{6545}\ln(\sqrt{2}+1)$$

$$+\frac{1}{7\sqrt{3}}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-1)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{37\pi\sqrt{3}}{4488}+\frac{3\pi\sqrt{2}}{1309}-\frac{607\pi}{39270}-\frac{9}{1309}\ln 3$$

$$+\frac{173}{39270}\ln 2+\frac{1}{20\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{4\sqrt{2}}{187}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)}$$

$$=\sum_{n=1}^{\infty}\Big[\frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)}$$

$$-\frac{2}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)}\Big]$$

$$=-\frac{351\pi\sqrt{3}}{7480}+\frac{32\pi\sqrt{2}}{2805}+\frac{129\pi}{1870}+\frac{2392}{19635}\ln 2$$

$$-\frac{972}{6545}\ln 3+\frac{40\sqrt{2}}{561}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)}$$

$$=-\frac{1}{8}\sum_{n=1}^{\infty}\Big[\frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)}$$

$$-\frac{3}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)}\Big]$$

$$=\frac{379\pi\sqrt{3}}{14960}-\frac{496\pi\sqrt{2}}{19635}-\frac{95\pi}{20944}-\frac{46}{561}\ln 2$$

$$+\frac{1107}{14960}\ln 3-\frac{856\sqrt{2}}{19635}\ln(\sqrt{2}+1)$$

$$+\frac{\sqrt{3}}{56}\ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\Big[\frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)}$$

$$-\frac{2}{(6n-1)(8n-7)(8n-5)(12n-11)(12n-7)}\Big]$$

$$=\frac{2341\pi\sqrt{3}}{157080}-\frac{144\pi\sqrt{2}}{6545}+\frac{146\pi}{19635}+\frac{432}{6545}\ln 3$$

$$-\frac{1354}{19635}\ln 2-\frac{152\sqrt{2}}{6545}\ln(\sqrt{2}+1)$$

$$+\frac{1}{70\sqrt{3}}\ln(2+\sqrt{3}).$$

7. $$\sum_{n=1}^{\infty}\Big[\frac{A}{k(A_1n-A_1)+a}-\frac{B}{k(B_1n-B_1)+b}$$

$$+\frac{D}{k(D_1n-D_1)+d}\Big]$$

定理15 设A、B、b、d都是正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(2n-2)+a}-\frac{3B}{k(3n-3)+b}+\frac{4B-2A}{k(4n-4)+d}\right]$$
$$=AC(a,2k)-3BC(b,3k)+(4B-2A)C(d,4k)$$
$$+\frac{1}{2k}\ln\frac{2^{4B-A}b^{2B}}{3^{2B}a^{A}d^{B-2A}};\qquad(1)$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(3n-3)+a}-\frac{2B}{k(4n-4)+b}+\frac{3B-2A}{k(6n-6)+d}\right]$$
$$=AC(a,3k)-2BC(b,4k)+(3B-2A)C(d,6k)$$
$$+\frac{1}{6k}\ln\frac{3^{2A}6^{3B-2A}b^{3B}}{4^{3B}a^{2A}d^{3B-2A}};\qquad(2)$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(2n-2)+a}-\frac{B}{k(3n-3)+b}+\frac{2B-3A}{k(6n-6)+d}\right]$$
$$=AC(a,2k)-BC(b,3k)+(2B-3A)C(d,6k)$$
$$+\frac{1}{6k}\ln\frac{2^{3A}6^{2B-3A}b^{2B}}{3^{2B}a^{3A}d^{2B-3A}};\qquad(3)$$

$$\sum_{n=1}^{\infty}\left[\frac{3A}{k(3n-3)+a}-\frac{3B}{k(6n-6)+b}+\frac{4B-8A}{k(8n-8)+d}\right]$$
$$=3AC(a,3k)-3BC(b,6k)+(4B-8A)C(d,8k)$$
$$+\frac{1}{2k}\ln\frac{3^{2A}8^{B-2A}b^{B}}{6^{B}a^{2A}d^{B-2A}};\qquad(4)$$

$$\sum_{n=1}^{\infty}\left[\frac{3A}{k(3n-3)+a}-\frac{B}{k(4n-4)+b}+\frac{2B-8A}{k(8n-8)+d}\right]$$
$$=3AC(a,3k)-BC(b,4k)+(2B-8A)C(d,8k)$$
$$+\frac{1}{4k}\ln\frac{3^{4A}2^{B-12A}b^{B}}{a^{4A}d^{B-4A}};\qquad(5)$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{2B}{k(2n-2)+b}+\frac{3B-3A}{k(3n-3)+d}\right]$$
$$=AC(a,k)-2BC(b,2k)+(3B-3A)C(d,3k)$$

$$+\frac{1}{k}\ln\frac{3^{B-A}b^{B}}{2^{B}a^{A}d^{B-A}};\qquad(6)$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(3n-3)+a}-\frac{2B}{k(2n-2)+b}+\frac{3B-A}{k(3n-3)+d}\right]$$

$$=AC(a,3k)-2BC(b,2k)+(3B-A)C(d,3k)$$

$$+\frac{1}{3k}\ln\frac{3^{3B}b^{3B}}{2^{3B}a^{A}d^{3B-A}};\qquad(7)$$

$$\sum_{n=1}^{\infty}\left[\frac{4A}{k(4n-4)+a}-\frac{B}{k(3n-3)+b}+\frac{B-3A}{k(3n-3)+d}\right]$$

$$=4AC(a,4k)-BC(b,3k)+(B-3A)C(d,3k)$$

$$+\frac{1}{3k}\ln\frac{4^{3A}3^{B-3A}b^{B}}{3^{B}a^{3A}d^{B-3A}};\qquad(8)$$

$$\sum_{n=1}^{\infty}\left[\frac{3A}{k(3n-3)+a}-\frac{B}{k(2n-2)+b}+\frac{B-2A}{k(2n-2)+d}\right]$$

$$=3AC(a,3k)-BC(b,2k)+(B-2A)C(d,2k)$$

$$+\frac{1}{2k}\ln\frac{3^{2A}2^{B-2A}b^{B}}{2^{B}a^{2A}d^{B-2A}};\qquad(9)$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(4n-4)+a}-\frac{3B}{k(3n-3)+b}+\frac{4B-A}{k(4n-4)+d}\right]$$

$$=AC(a,4k)-3BC(b,3k)+(4B-A)C(d,4k)$$

$$+\frac{1}{4k}\ln\frac{4^{4B}b^{4B}}{3^{4B}a^{A}d^{4B-A}}.\qquad(10)$$

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{A}{k(2i-2)+a}-\frac{3B}{k(3i-3)+b}+\frac{4B-2A}{k(4i-4)+d}\right]$$

$$=AC(a,2k)-3BC(b,3k)+(4B-2A)C(d,4k)$$

$$+\frac{1}{12k}\ln\frac{(2nk-2k+a)^{6A}(4nk-4k+d)^{12B-6A}}{(3nk-3k+b)^{12B}}$$

$$+\frac{1}{12k}\ln\frac{b^{12B}}{a^{6A}d^{12B-6A}}+A\varepsilon_n(a,2k)-3B\varepsilon_n(b,3k)$$

$$+(4B-2A)\varepsilon_n(d,4k).$$

令 $n\to\infty$并取极限，即得（1）.

仿此可证定理的其余各部分.

例1 求和：

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(3n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)};$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-2)(3n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(2n-1)(3n-2)(3n-1)}.$$

解 用分项分式方法求得

$$\frac{1}{(4n-3)(6n-5)(6n-4)}=\frac{3}{6n-5}-\frac{4}{4n-3}+\frac{3}{6n-4}.$$

于定理15（7）令 $A=3$，$B=2$，$a=1$，$b=1$，$d=2$，$k=2$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{3}{6n-5}-\frac{4}{4n-3}+\frac{3}{6n-4}\right)$$

$$=3C(1,6)-4C(1,4)+\frac{3}{2}C(1,3)+\frac{1}{2}\ln\frac{3^2}{2^3}.$$

根据定理6，有

$$3C(1,6)-2C(1,4)=\frac{\pi\sqrt{3}}{4}-\frac{\pi}{4}+\frac{1}{4}\ln 3,$$

$$\frac{3}{2}C(1,3)-2C(1,4)=\frac{\pi}{4\sqrt{3}}-\frac{\pi}{4}+\frac{1}{4}\ln\frac{3}{2^2}.$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$=\frac{\pi}{\sqrt{3}}-\frac{\pi}{2}+\frac{3}{2}\ln 3-2\ln 2.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-1)(6n-2)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{3}{6n-2}-\frac{4}{4n-1}+\frac{3}{6n-1}\right)$$

$$=-\frac{\pi}{\sqrt{3}}+\frac{\pi}{2}+\frac{3}{2}\ln 3-2\ln 2.$$

用分项分式方法求得

$$\frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$=\frac{1}{2}\left(\frac{5}{6n-5}-\frac{6}{4n-3}+\frac{4}{6n-4}\right).$$

于定理15(7)令 $A=5$, $B=3$, $a=1$, $b=1$, $d=2$, $k=2$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{5}{6n-5}-\frac{6}{4n-3}+\frac{4}{6n-4}\right)$$

$$=\frac{5}{2}C(1,6)-3C(1,4)+C(1,3)+\frac{1}{12}\ln\frac{3^9}{2^{13}}.$$

根据定理6，有

$$\frac{5}{2}C(1,6)-\frac{5}{3}C(1,4)$$

$$=\frac{5\pi}{8\sqrt{3}}-\frac{5\pi}{24}+\frac{1}{24}\ln 3^{5},$$

$$C(1,3)-\frac{4}{3}C(1,4)$$

$$=\frac{\pi}{6\sqrt{3}}-\frac{\pi}{6}+\frac{1}{6}\ln\frac{3}{2^{2}}.$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-5)(6n-4)}$

$$=\frac{19\pi}{24\sqrt{3}}-\frac{3\pi}{8}+\frac{9}{8}\ln 3-\frac{17}{12}\ln 2.$$

仿此求得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-2)(6n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{6n-2}-\frac{2}{4n-1}+\frac{1}{6n-1}\right)$$

$$=-\frac{5\pi}{24\sqrt{3}}+\frac{\pi}{8}+\frac{3}{8}\ln 3-\frac{7}{12}\ln 2.$$

从而 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-2)(3n-1)}$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)}\right.$$

$$\left.+\frac{1}{(4n-1)(6n-2)(6n-1)}\right]$$

$$=3\ln 3-4\ln 2,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)(3n-2)(3n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)(6n-5)(6n-4)}\right.$$

$$-\frac{1}{(4n-1)(6n-2)(6n-1)}\Bigg]$$

$$=\frac{2\pi}{\sqrt{3}}-\pi,$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-2)(3n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(4n-3)(6n-5)(6n-4)}$$

$$+\frac{2n}{(4n-1)(6n-2)(6n-1)}\Bigg]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$+2\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-2)(6n-1)}$$

$$=\frac{\pi}{6\sqrt{3}}+\frac{3}{2}\ln 3-2\ln 2,$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(2n-1)(3n-2)(3n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(4n-3)(6n-5)(6n-4)}$$

$$-\frac{2n}{(4n-1)(6n-2)(6n-1)}\Bigg]$$

$$=\frac{\pi}{\sqrt{3}}-\frac{\pi}{2}+\frac{1}{3}\ln 2.$$

例2 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(4n-1)(6n-1)(8n+1)}.$$

解 用分项分式方法求得

$$\frac{n}{(3n-2)(4n-1)(6n-1)}$$

$$=\frac{1}{15}\left(\frac{2}{3n-2}-\frac{6}{4n-1}+\frac{5}{6n-1}\right).$$

于定理15(2)令 $A=2$, $B=3$, $a=1$, $b=3$, $d=5$, $k=1$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(4n-1)(6n-1)}$$

$$=\frac{1}{15}\sum_{n=1}^{\infty}\left(\frac{2}{3n-2}-\frac{6}{4n-1}+\frac{5}{6n-1}\right)$$

$$=\frac{2}{15}C(1,3)-\frac{2}{5}C(3,4)+\frac{1}{3}C(5,6)$$

$$+\frac{1}{90}\ln\frac{3^{18}}{2^{18}5^{5}}.$$

根据定理6，有

$$\frac{2}{15}C(1,3)-\frac{8}{45}C(3,4)$$

$$=\frac{\pi}{45\sqrt{3}}+\frac{\pi}{45}-\frac{1}{45}\ln 2^{2}\cdot 3,$$

$$\frac{1}{3}C(5,6)-\frac{2}{9}C(3,4)$$

$$=-\frac{\pi}{12\sqrt{3}}+\frac{\pi}{36}+\frac{1}{36}\ln\frac{5^{2}}{3}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(4n-6)(6n-1)}$$

$$=-\frac{11\pi}{180\sqrt{3}}+\frac{\pi}{20}+\frac{3}{20}\ln 3-\frac{17}{90}\ln 2.$$

用分项分式方法求得

$$\frac{n}{(4n-1)(6n-1)(8n+1)}$$

$$=\frac{1}{42}\left(\frac{7}{4n-1}-\frac{9}{6n-1}-\frac{2}{8n+1}\right).$$

于定理15(1)令 $A=7$, $B=3$, $a=3$, $b=5$, $d=9$, $k=2$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n+1)}$$

$$=\frac{1}{42}\sum_{n=1}^{\infty}\left(\frac{7}{4n-1}-\frac{9}{6n-1}-\frac{2}{8n+1}\right)$$

$$=\frac{1}{6}C(3,4)-\frac{3}{14}C(5,6)-\frac{1}{21}C(9,8)$$

$$+\frac{1}{168}\ln\frac{2^5 5^6}{3^{11}}.$$

于命题2令 $a=1$, $b=1$, $k=8$, 则有

$$C(9,8)=C(1,8)+\frac{1}{8}\ln 3^2-1,$$

$$-\frac{1}{21}C(9,8)=-\frac{1}{21}C(1,8)-\frac{1}{168}\ln 3^2+\frac{1}{21}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n+1)}$$

$$=\frac{1}{6}C(3,4)-\frac{3}{14}C(5,6)-\frac{1}{21}C(1,8)$$

$$+\frac{1}{168}\ln\frac{2^5 5^6}{3^{13}}+\frac{1}{21}.$$

根据定理6，有

$$\frac{1}{7}C(3,4)-\frac{3}{14}C(5,6)$$

$$=\frac{\pi\sqrt{3}}{56}-\frac{\pi}{56}+\frac{1}{56}\ln\frac{3}{5^2},$$

$$\frac{1}{42}C(3,4)-\frac{1}{21}C(1,8)$$

$$=-\frac{\pi}{168\sqrt{2}}-\frac{\pi}{168}+\frac{1}{168}\ln 3$$

$$-\frac{1}{84\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $\sum\limits_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n+1)}$

$$=\frac{\pi\sqrt{3}}{56}-\frac{\pi}{168\sqrt{2}}-\frac{\pi}{42}+\frac{5}{168}\ln 2-\frac{3}{56}\ln 3$$

$$-\frac{1}{84\sqrt{2}}\ln(\sqrt{2}+1)+\frac{1}{21}.$$

从而 $\sum\limits_{n=1}^{\infty}\frac{n}{(3n-2)(4n-1)(6n-1)(8n+1)}$

$$=\frac{1}{19}\sum_{n=1}^{\infty}\left[\frac{3n}{(3n-2)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{8n}{(4n-1)(6n-1)(8n+1)}\right]$$

$$=-\frac{257\pi\sqrt{3}}{23940}+\frac{\pi}{399\sqrt{2}}+\frac{143\pi}{7980}+\frac{123}{2660}\ln 3$$

$$-\frac{169}{3990}\ln 2+\frac{\sqrt{2}}{399}\ln(\sqrt{2}+1)-\frac{8}{399}.$$

例3 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 用分项分式方法求得

$$\frac{n}{(2n-1)(3n-1)(4n-1)}=\frac{1}{2n-1}-\frac{3}{3n-1}+\frac{2}{4n-1}.$$

于定理15(1)令 $A=1$, $B=1$, $a=1$, $b=2$, $d=3$, $k=1$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{3}{3n-1}+\frac{2}{4n-1}\right)$$

$$=C(1,2)-3C(2,3)+2C(3,4)+\frac{1}{2}\ln\frac{2^5}{3^3}.$$

根据定理6，有

$$C(1,2)-\frac{3}{2}C(2,3)=\frac{\pi}{4\sqrt{3}}-\frac{1}{4}\ln 3,$$

$$2C(3,4)-\frac{3}{2}C(2,3)=\frac{\pi}{4\sqrt{3}}-\frac{\pi}{4}+\frac{1}{4}\ln 3.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)}$$

$$=\frac{\pi}{2\sqrt{3}}-\frac{\pi}{4}+\frac{5}{2}\ln 2-\frac{3}{2}\ln 3.$$

于定理15(2)令 $A=1$, $B=1$, $a=2$, $b=3$, $d=5$, $k=1$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{3n-1}-\frac{3}{4n-1}+\frac{2}{6n-1}\right)$$

$$=-\frac{5\pi}{12\sqrt{3}}+\frac{\pi}{4}+\frac{3}{4}\ln 3-\frac{7}{6}\ln 2;$$

于定理15(1)令 $A=1$, $B=1$, $a=3$, $b=5$, $d=7$, $k=2$,

即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{3}{6n-1}+\frac{2}{8n-1}\right)$$

$$=\frac{\pi\sqrt{3}}{8}-\frac{\pi}{8\sqrt{2}}-\frac{\pi}{8}+\frac{3}{8}\ln 2-\frac{3}{8}\ln 3$$

$$+\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理 15(2)令 $A=1$, $B=1$, $a=5$, $b=7$, $d=11$, $k=2$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{6n-1}-\frac{2}{8n-1}+\frac{1}{12n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{16}+\frac{\pi}{8\sqrt{2}}+\frac{\pi}{48}+\frac{3}{16}\ln 3-\frac{5}{24}\ln 2$$

$$+\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)}\right.$$

$$\left.-\frac{3n}{(3n-1)(4n-1)(6n-1)}\right]$$

$$=\frac{7\pi}{8\sqrt{3}}-\frac{\pi}{2}+3\ln 2-\frac{15}{8}\ln 3,$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3n}{(3n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{8n}{(4n-1)(6n-1)(8n-1)}\right]$$

$$=-\frac{17\pi}{20\sqrt{3}}+\frac{\pi}{5\sqrt{2}}+\frac{7\pi}{20}+\frac{21}{20}\ln 3-\frac{13}{10}\ln 2$$

$$-\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{n}{(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{3n}{(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{5\pi\sqrt{3}}{32}-\frac{\pi}{4\sqrt{2}}-\frac{3\pi}{32}+\frac{1}{2}\ln 2-\frac{15}{32}\ln 3$$

$$-\frac{\sqrt{3}}{16}\ln(2+\sqrt{3})+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{4n}{(3n-1)(4n-1)(6n-1)(8n-1)}\right]$$

$$=\frac{19\pi\sqrt{3}}{40}-\frac{2\pi\sqrt{2}}{15}-\frac{19\pi}{30}+\frac{41}{15}\ln 2-\frac{81}{40}\ln 3$$

$$+\frac{4\sqrt{2}}{15}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(3n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{4n}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{109\pi}{120\sqrt{3}}+\frac{\pi\sqrt{2}}{5}+\frac{29\pi}{120}+\frac{39}{40}\ln 3-\frac{11}{10}\ln 2$$

$$-\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1)+\frac{1}{4\sqrt{3}}\ln(2+\sqrt{3}).$$

从而

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6n}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{11\pi}{8\sqrt{3}}-\frac{4\pi\sqrt{2}}{15}-\frac{5\pi}{12}+\frac{28}{15}\ln 2-\frac{63}{40}\ln 3$$

$$+\frac{8\sqrt{2}}{15}\ln(\sqrt{2}+1)-\frac{\sqrt{3}}{10}\ln(2+\sqrt{3}).$$

例4 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理15（6）令 $A=1, B=2, a=1$, $b=3$, $d=5$, $k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(4n-1)(6n-1)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{4}{4n-1}+\frac{3}{6n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{16}+\frac{\pi}{8}+\frac{3}{16}\ln 3-\frac{1}{4}\ln 2.$$

由此以及例3中的相应结果即可顺次推得

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{4n}{(4n-1)(6n-1)(8n-1)}\right]$$

$$=-\frac{3\pi\sqrt{3}}{16}+\frac{\pi}{6\sqrt{2}}+\frac{5\pi}{24}+\frac{9}{16}\ln 3-\frac{7}{12}\ln 2$$

$$-\frac{1}{3\sqrt{2}}\ln(\sqrt{2}+1).$$

从而 $$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6n}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{9\pi\sqrt{3}}{40}+\frac{\pi}{3\sqrt{2}}+\frac{37\pi}{240}+\frac{27}{40}\ln 3-\frac{43}{60}\ln 2$$

$$+\frac{3\sqrt{3}}{40}\ln(2+\sqrt{3})-\frac{\sqrt{2}}{3}\ln(\sqrt{2}+1).$$

例5 求和:

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理15(3)令 $A=1$, $B=2$, $a=1$, $b=2$, d

$=5$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(6n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{2}{3n-1}+\frac{1}{6n-1}\right)$$

$$=\frac{\pi}{24\sqrt{3}}+\frac{2}{3}\ln 2-\frac{3}{8}\ln 3,$$

于定理15(4)令 $A=1$，$B=5$，$a=2$，$b=5$，$d=7$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{15}\sum_{n=1}^{\infty}\left(\frac{3}{3n-1}-\frac{15}{6n-1}+\frac{12}{8n-1}\right)$$

$$=\frac{13\pi}{60\sqrt{3}}-\frac{\pi}{10\sqrt{2}}-\frac{\pi}{20}+\frac{1}{15}\ln 2-\frac{3}{20}\ln 3$$

$$+\frac{1}{5\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{4n}{(2n-1)(3n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(6n-1)}\right.$$

$$\left.-\frac{4n}{(3n-1)(6n-1)(8n-1)}\right]$$

$$=-\frac{11\pi}{40\sqrt{3}}+\frac{\pi\sqrt{2}}{15}+\frac{\pi}{15}+\frac{2}{15}\ln 2+\frac{3}{40}\ln 3$$

$$-\frac{2\sqrt{2}}{15}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(3n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{4n}{(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{29\pi}{90\sqrt{3}}-\frac{\pi}{5\sqrt{2}}-\frac{2\pi}{45}+\frac{3}{10}\ln 2-\frac{3}{10}\ln 3$$

$$+\frac{\sqrt{2}}{5}\ln(\sqrt{2}+1)-\frac{1}{6\sqrt{3}}\ln(2+\sqrt{3}).$$

从而 $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6n}{(3n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{53\pi}{120\sqrt{3}}+\frac{2\pi\sqrt{2}}{15}+\frac{\pi}{15}-\frac{1}{3}\ln 2+\frac{3}{8}\ln 3$$

$$-\frac{4\sqrt{2}}{15}\ln(\sqrt{2}+1)+\frac{1}{5\sqrt{3}}\ln(2+\sqrt{3}).$$

例6 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}.$$

解 于定理15(5)令 $A=1$，$B=5$，$a=2$，$b=3$，$d=7$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(8n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{3}{3n-1}-\frac{5}{4n-1}+\frac{2}{8n-1}\right)$$

$$=-\frac{\pi}{10\sqrt{3}}-\frac{\pi}{20\sqrt{2}}+\frac{\pi}{10}+\frac{3}{10}\ln 3-\frac{11}{20}\ln 2$$

$$+\frac{1}{10\sqrt{2}}\ln(\sqrt{2}+1),$$

于定理15(6)令 $A=1,\ B=2,\ a=3,\ b=7,\ d=11, k=4,$ 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{4}{8n-1}+\frac{3}{12n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{64}+\frac{\pi}{16\sqrt{2}}-\frac{\pi}{64}+\frac{3}{64}\ln 3-\frac{1}{16}\ln 2$$

$$+\frac{\sqrt{3}}{32}\ln(2+\sqrt{3})-\frac{1}{8\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)}\right.$$

$$\left.-\frac{4n}{(3n-1)(4n-1)(8n-1)}\right]$$

$$=\frac{\pi\sqrt{3}}{10}+\frac{\pi}{15\sqrt{2}}-\frac{13\pi}{60}+\frac{47}{30}\ln 2-\frac{9}{10}\ln 3$$

$$-\frac{\sqrt{2}}{15}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(3n-1)(4n-1)(8n-1)}\right.$$

$$\left.-\frac{4n}{(4n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{7\pi}{240\sqrt{3}}-\frac{\pi}{10\sqrt{2}}+\frac{13\pi}{240}+\frac{3}{80}\ln 3-\frac{1}{10}\ln 2$$

$$+\frac{1}{5\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3}).$$

从而 $$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)}\right.$$

$$\left.-\frac{6n}{(3n-1)(4n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{\pi}{40\sqrt{3}}+\frac{\pi\sqrt{2}}{15}-\frac{13\pi}{120}+\frac{13}{30}\ln 2-\frac{9}{40}\ln 3$$

$$-\frac{2\sqrt{2}}{15}\ln(\sqrt{2}+1)+\frac{\sqrt{3}}{20}\ln(2+\sqrt{3}).$$

例7 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}.$$

解 于定理15(3)令 $A=1, B=2, a=3, b=5, d=11$, $k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{2}{6n-1}+\frac{1}{12n-1}\right)$$

$$=\frac{\pi\sqrt{3}}{32}-\frac{5\pi}{96}+\frac{1}{12}\ln 2-\frac{3}{32}\ln 3$$

$$+\frac{1}{16\sqrt{3}}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(3n-1)(4n-1)(6n-1)}\right.$$

$$-\frac{4n}{(4n-1)(6n-1)(12n-1)}\Bigg]$$

$$=-\frac{19\pi}{72\sqrt{3}}+\frac{11\pi}{72}+\frac{3}{8}\ln 3-\frac{1}{2}\ln 2$$

$$-\frac{1}{12\sqrt{3}}\ln(2+\sqrt{3}),$$

从而 $$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\Bigg[\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$-\frac{6n}{(3n-1)(4n-1)(6n-1)(12n-1)}\Bigg]$$

$$=\frac{59\pi}{120\sqrt{3}}-\frac{17\pi}{60}+\frac{6}{5}\ln 2-\frac{33}{40}\ln 3$$

$$+\frac{1}{10\sqrt{3}}\ln(2+\sqrt{3}).$$

例8 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(2n-1)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-1)(4n-1)(6n-1)},$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}.$$

解 于定理15(1)令 $A=3, B=4$, $a=1$, $b=2, d=2$, $k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-4)(8n-5)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{4n-3}-\frac{12}{6n-4}+\frac{10}{8n-5}\right)$$

$$=-\frac{\pi}{2\sqrt{3}}+\frac{5\pi}{8\sqrt{2}}-\frac{\pi}{8}+\frac{29}{8}\ln 2-\frac{3}{2}\ln 2$$

$$-\frac{5}{4\sqrt{2}}\ln(\sqrt{2}+1),$$

于定理15(2)令 $A=4$, $B=5$, $a=2$, $b=5$, $d=5$, $k=2$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(8n-5)(12n-7)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{6n-4}-\frac{10}{8n-5}+\frac{7}{12n-7}\right)$$

$$=-\frac{13\pi}{48\sqrt{3}}-\frac{5\pi}{8\sqrt{2}}+\frac{29\pi}{48}+\frac{15}{16}\ln 3-\frac{13}{8}\ln 2$$

$$-\frac{7}{8\sqrt{3}}\ln(2+\sqrt{3})+\frac{5}{4\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{n}{(4n-3)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{3n}{(6n-4)(8n-5)(12n-7)}\right]$$

$$=\frac{5\pi}{32\sqrt{3}}+\frac{5\pi}{4\sqrt{2}}-\frac{31\pi}{32}+\frac{17}{4}\ln 2-\frac{69}{32}\ln 3$$

$$+\frac{7\sqrt{3}}{16}\ln(2+\sqrt{3})-\frac{5}{2\sqrt{2}}\ln(\sqrt{2}+1).$$

由此以及上面已经求得的相应的结果，即可推得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(2n-1)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(4n-3)(6n-4)(8n-5)}\right.$$

$$
-\frac{2n}{(4n-1)(6n-1)(8n-1)}\Bigg]
$$

$$
=2\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-4)(8n-5)}
$$

$$
-\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)(8n-5)}
$$

$$
-2\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n-1)}
$$

$$
=-\frac{\pi}{\sqrt{3}}+\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\ln 2-\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1),
$$

$$
\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-1)(4n-1)(6n-1)}
$$

$$
=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(6n-4)(8n-5)(12n-7)}
$$

$$
-\frac{2n}{(6n-1)(8n-1)(12n-1)}\Bigg]
$$

$$
=\frac{\pi}{3\sqrt{3}}-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{6}-\frac{1}{3}\ln 2+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)
$$

$$
-\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3}),
$$

$$
\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}
$$

$$
=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(4n-3)(6n-4)(8n-5)(12n-7)}
$$

$$
-\frac{2n}{(4n-1)(6n-1)(8n-1)(12n-1)}\Bigg]
$$

$$
=2\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-4)(8n-5)(12n-7)}
$$

$$
-\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)}
$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=-\frac{\pi}{\sqrt{3}}+\frac{\pi}{\sqrt{2}}-\frac{\pi}{8}+\ln 2+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3})$$

$$-\sqrt{2}\ln(\sqrt{2}+1).$$

仿此求得以下一系列的结果。

于定理15(8)令 $A=7$，$B=25$，$a=1$，$b=1$，$d=2$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left(\frac{28}{8n-7}-\frac{25}{6n-5}+\frac{4}{6n-4}\right)$$

$$=-\frac{71\pi}{120\sqrt{3}}+\frac{7\pi}{20\sqrt{2}}+\frac{7\pi}{40}+\frac{17}{30}\ln 2-\frac{21}{40}\ln 3$$

$$+\frac{7}{10\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理 15(10) 令 $A=7$，$B=8$，$a=1$，$b=2$，$d=3$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(8n-7)(8n-5)}$$

$$=\frac{1}{20}\sum_{n=1}^{\infty}\left(\frac{4}{8n-7}-\frac{24}{6n-4}+\frac{25}{8n-5}\right)$$

$$=-\frac{\pi}{10\sqrt{3}}+\frac{\pi}{5\sqrt{2}}-\frac{9\pi}{160}+\frac{4}{5}\ln 2-\frac{3}{10}\ln 3$$

$$-\frac{9}{40\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理15(9)令 $A=22$，$B=49$，$a=1$，$b=1$，$d=3$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(12n-11)}$$

$$=\frac{1}{56}\sum_{n=1}^{\infty}\left(\frac{66}{12n-11}-\frac{49}{8n-7}+\frac{5}{8n-5}\right)$$

$$=\frac{11\pi\sqrt{3}}{224}-\frac{11\pi}{112\sqrt{2}}+\frac{17\pi}{448}+\frac{33}{224}\ln 3-\frac{11}{112}\ln 2$$

$$+\frac{11\sqrt{3}}{112}\ln(2+\sqrt{3})-\frac{27}{112\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理15(7)令 $A=11$，$B=20$，$a=1$，$b=3$，$d=5$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{112}\sum_{n=1}^{\infty}\left(\frac{11}{12n-11}-\frac{40}{8n-5}+\frac{49}{12n-7}\right)$$

$$=-\frac{19\pi}{448\sqrt{3}}-\frac{5\pi}{112\sqrt{2}}+\frac{15\pi}{224}+\frac{15}{224}\ln 3-\frac{5}{112}\ln 2$$

$$-\frac{19}{224\sqrt{3}}\ln(2+\sqrt{3})+\frac{5}{56\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理15(8)令 $A=3$，$B=10$，$a=5$，$b=4$，$d=5$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)}$$

$$=\frac{1}{10}\sum_{n=1}^{\infty}\left(\frac{12}{8n-3}-\frac{10}{6n-2}+\frac{1}{6n-1}\right)$$

$$=\frac{7\pi}{120\sqrt{3}}-\frac{3\pi}{20\sqrt{2}}+\frac{3\pi}{40}+\frac{19}{30}\ln 2-\frac{9}{40}\ln 3$$

$$-\frac{3}{10\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理 15(10) 令 $A=3$, $B=2$, $a=5$, $b=5$, $d=7$, $k=2$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(6n-1)(8n-3)(8n-1)}$$

$$=\frac{1}{20}\sum_{n=1}^{\infty}\left(\frac{3}{8n-3}-\frac{6}{6n-1}+\frac{5}{8n-1}\right)$$

$$=\frac{\pi\sqrt{3}}{40}-\frac{\pi}{20\sqrt{2}}-\frac{\pi}{160}+\frac{1}{10}\ln 2-\frac{3}{40}\ln 3$$

$$+\frac{1}{40\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理15(9)令$A=10$, $B=21$, $a=7$, $b=5$, $d=7$, $k=4$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(8n-3)(8n-1)(12n-5)}$$

$$=\frac{1}{56}\sum_{n=1}^{\infty}\left(\frac{30}{12n-5}-\frac{21}{8n-3}+\frac{1}{8n-1}\right)$$

$$=\frac{5\pi\sqrt{3}}{224}+\frac{5\pi}{112\sqrt{2}}-\frac{31\pi}{448}+\frac{15}{224}\ln 3-\frac{5}{112}\ln 2$$

$$-\frac{5\sqrt{3}}{112}\ln(2+\sqrt{3})+\frac{11}{112\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理15(7)令 $A=5$, $B=4$, $a=7$, $b=7$, $d=11$, $k=4$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{112}\sum_{n=1}^{\infty}\left(\frac{5}{12n-5}-\frac{8}{8n-1}+\frac{7}{12n-1}\right)$$

$$=-\frac{\pi}{448\sqrt{3}}+\frac{\pi}{112\sqrt{2}}-\frac{\pi}{224}+\frac{3}{224}\ln 3-\frac{1}{112}\ln 2$$

$$+\frac{1}{224\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{56\sqrt{2}}\ln(\sqrt{2}+1).$$

由此以及上面已求得的相应结果即可顺次推得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-2)(3n-1)(4n-3)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)}\right.$$

$$\left.-\frac{2n}{(6n-2)(6n-1)(8n-3)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-7)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)}$$

$$=-\frac{\pi\sqrt{3}}{5}+\frac{3\pi}{5\sqrt{2}}-\frac{11}{15}\ln 2+\frac{3\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(3n-1)(4n-3)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)}\right.$$

$$\left.+\frac{2n}{(6n-2)(6n-1)(8n-3)}\right]$$

$$=-\frac{11\pi}{30\sqrt{3}}+\frac{3\pi}{10}+\frac{9}{5}\ln 2-\frac{9}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-4)(8n-7)(8n-5)}\right.$$

$$-\frac{2n}{(6n-1)(8n-3)(8n-1)}\Bigg]$$

$$=-\frac{\pi}{5\sqrt{3}}+\frac{\pi}{5\sqrt{2}}+\frac{1}{5}\ln 2-\frac{1}{10\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(6n-4)(8n-7)(8n-5)}$$

$$+\frac{2n}{(6n-1)(8n-3)(8n-1)}\Bigg]$$

$$=\frac{\pi}{10\sqrt{3}}-\frac{\pi}{40}+\frac{3}{5}\ln 2-\frac{3}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(8n-7)(8n-5)(12n-11)}$$

$$-\frac{2n}{(8n-3)(8n-1)(12n-5)}\Bigg]$$

$$=-\frac{5\pi}{28\sqrt{2}}+\frac{5\pi}{28}+\frac{5\sqrt{3}}{28}\ln(2+\sqrt{3})$$

$$-\frac{11}{28\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(8n-7)(8n-5)(12n-11)}$$

$$+\frac{2n}{(8n-3)(8n-1)(12n-5)}\Bigg]$$

$$=\frac{5\pi\sqrt{3}}{56}-\frac{11\pi}{112}+\frac{15}{56}\ln 3-\frac{5}{28}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{2n}{(8n-1)(12n-5)(12n-1)}\right]$$

$$=-\frac{\pi\sqrt{2}}{56}+\frac{\pi}{28}-\frac{1}{56\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{1}{14\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{2n}{(8n-1)(12n-5)(12n-1)}\right]$$

$$=-\frac{\pi}{112\sqrt{3}}+\frac{\pi}{56}+\frac{3}{56}\ln 3-\frac{1}{28}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3n}{(6n-5)(6n-4)(8n-7)}\right.$$

$$\left.-\frac{4n}{(6n-4)(8n-7)(8n-5)}\right]$$

$$=-\frac{11\pi}{40\sqrt{3}}+\frac{\pi}{20\sqrt{2}}+\frac{3\pi}{20}-\frac{3}{10}\ln 2-\frac{3}{40}\ln 3$$

$$+\frac{3}{5\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$=-\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{2n}{(8n-7)(8n-5)(12n-11)}\right]$$

$$=\frac{221\pi}{1680\sqrt{3}}-\frac{37\pi}{280\sqrt{2}}+\frac{37\pi}{840}+\frac{111}{560}\ln 3-\frac{93}{280}\ln 2$$

$$+\frac{11}{56\sqrt{3}}\ln(2+\sqrt{3})-\frac{3}{35\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{2n}{(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{3n}{(8n-5)(12n-11)(12n-7)}\right]$$

$$=\frac{9\pi\sqrt{3}}{448}-\frac{\pi}{112\sqrt{2}}-\frac{\pi}{56}+\frac{3}{224}\ln 3-\frac{1}{112}\ln 2$$

$$+\frac{9\sqrt{3}}{224}\ln(2+\sqrt{3})-\frac{3}{28\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{3n}{(6n-2)(6n-1)(8n-3)}\right.$$

$$\left.-\frac{4n}{(6n-1)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi}{40\sqrt{3}}-\frac{\pi}{20\sqrt{2}}+\frac{\pi}{20}+\frac{3}{10}\ln 2-\frac{3}{40}\ln 3$$

$$-\frac{1}{5\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$=-\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(6n-1)(8n-3)(8n-1)}\right.$$

$$\left.-\frac{2n}{(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{11\pi}{560\sqrt{3}}+\frac{13\pi}{280\sqrt{2}}-\frac{37\pi}{840}+\frac{39}{560}\ln 3-\frac{53}{840}\ln 2$$

$$-\frac{5}{56\sqrt{3}}\ln(2+\sqrt{3})+\frac{\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{7}\sum_{n=1}^{\infty}\left[\frac{2n}{(8n-3)(8n-1)(12n-5)}\right.$$

$$\left.-\frac{3n}{(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{3\pi\sqrt{3}}{448}+\frac{\pi}{112\sqrt{2}}-\frac{\pi}{56}+\frac{3}{224}\ln 3-\frac{1}{112}\ln 2$$

$$-\frac{3\sqrt{3}}{224}\ln(2+\sqrt{3})+\frac{1}{28\sqrt{2}}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)}$$

$$=-\frac{\pi}{5\sqrt{3}}+\frac{\pi}{5\sqrt{2}}-\frac{3}{5}\ln 2+\frac{2\sqrt{2}}{5}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.+\frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)}\right]$$

$$=-\frac{\pi\sqrt{3}}{10}+\frac{\pi}{5}+\frac{3}{5}\ln 2-\frac{3}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{\pi}{15\sqrt{3}}-\frac{13\pi}{70\sqrt{2}}+\frac{5\pi}{42}-\frac{1}{15}\ln 2$$

$$+\frac{5}{14\sqrt{8}}\ln(2+\sqrt{3})-\frac{4\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.+\frac{2n}{(6n-1)(8n-3)(8n-1)(12n-11)}\right]$$

$$=\frac{61\pi}{420\sqrt{3}}-\frac{2\pi}{35}+\frac{39}{140}\ln 3-\frac{67}{210}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{2n}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=-\frac{\pi}{28\sqrt{2}}+\frac{\pi}{28}+\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3})$$

$$-\frac{1}{7\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{2n}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{3\pi\sqrt{3}}{112}-\frac{\pi}{28}+\frac{3}{56}\ln 3-\frac{1}{28}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$=-\sum_{n=1}^{\infty}\left[\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{2n}{(6n-4)(8n-7)(8n-5)(12n-11)}\right]$$

$$=\frac{113\pi}{210\sqrt{3}}-\frac{11\pi}{35\sqrt{2}}-\frac{13\pi}{210}+\frac{33}{70}\ln 3-\frac{51}{140}\ln 2$$

$$+\frac{11}{28\sqrt{3}}\ln(2+\sqrt{3})-\frac{27}{35\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{n}{(6n-4)(8n-7)(8n-5)(12n-11)}-\frac{2n}{(8n-7)(8n-5)(12n-11)(12n-7)}\right]$$

$$=\frac{37\pi}{3360\sqrt{3}}-\frac{2\pi\sqrt{2}}{35}+\frac{67\pi}{840}+\frac{6}{35}\ln 3-\frac{11}{35}\ln 2-\frac{5}{112\sqrt{3}}\ln(2+\sqrt{3})+\frac{9}{70\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$=-\sum_{n=1}^{\infty}\left[\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)}-\frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{3\pi\sqrt{3}}{140}+\frac{\pi}{7\sqrt{2}}-\frac{29\pi}{210}+\frac{3}{14}\ln 3-\frac{179}{420}\ln 2-\frac{5}{28\sqrt{3}}\ln(2+\sqrt{3})+\frac{11}{35\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{n}{(6n-1)(8n-3)(8n-1)(12n-5)}-\frac{2n}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=-\frac{23\pi}{1120\sqrt{3}}+\frac{\pi}{35\sqrt{2}}-\frac{\pi}{120}+\frac{3}{70}\ln 3-\frac{19}{420}\ln 2-\frac{1}{112\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{70\sqrt{2}}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$=\frac{\pi}{3\sqrt{3}}-\frac{2\pi\sqrt{2}}{7}+\frac{5\pi}{21}+\frac{7}{15}\ln 2$$

$$+\frac{5}{7\sqrt{3}}\ln(2+\sqrt{3})-\frac{22\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.+\frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{62\pi}{105\sqrt{3}}-\frac{11\pi}{35}+\frac{6}{7}\ln 3-\frac{26}{21}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{\pi}{15\sqrt{3}}-\frac{2\pi\sqrt{2}}{35}+\frac{\pi}{21}-\frac{1}{15}\ln 2$$

$$+\frac{1}{28\sqrt{3}}\ln(2+\sqrt{3})+\frac{\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=-\frac{13\pi}{840\sqrt{3}}+\frac{\pi}{70}+\frac{6}{35}\ln 3-\frac{26}{105}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{2n}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right]$$

$$=\frac{289\pi}{1680\sqrt{3}}-\frac{\pi}{35\sqrt{2}}-\frac{31\pi}{420}+\frac{3}{70}\ln 3+\frac{37}{420}\ln 2$$

$$+\frac{3\sqrt{3}}{56}\ln(2+\sqrt{3})-\frac{6\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right.$$

$$-\frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{59\pi}{1680\sqrt{3}}+\frac{\pi}{35\sqrt{2}}-\frac{17\pi}{420}+\frac{3}{70}\ln 3-\frac{47}{420}\ln 2$$

$$-\frac{\sqrt{3}}{56}\ln(2+\sqrt{3})+\frac{2\sqrt{2}}{35}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$-\frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=2\sum_{n=1}^{\infty}\frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$-\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$-2\sum_{n=1}^{\infty}\frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{\pi}{15\sqrt{3}}-\frac{2\pi\sqrt{2}}{35}+\frac{\pi}{21}+\frac{1}{5}\ln 2+\frac{\sqrt{3}}{14}\ln(2+\sqrt{3})$$

$$-\frac{8\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$
=\sum_{n=1}^{\infty}\left[\frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.
$$

$$
\left.+\frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]
$$

$$
=\frac{87\pi}{420\sqrt{3}}-\frac{4\pi}{35}+\frac{6}{35}\ln 3-\frac{26}{105}\ln 2.
$$

8. $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\mathbf{A}}{\boldsymbol{k}(\mathbf{A}_1\boldsymbol{n}-\mathbf{A}_1)+\boldsymbol{a}}-\frac{\mathbf{B}}{\boldsymbol{k}(\mathbf{B}_1\boldsymbol{n}-\mathbf{B}_1)+\boldsymbol{b}}\right.$

$$
\left.+\frac{\mathbf{D}}{\boldsymbol{k}(\mathbf{D}_1\boldsymbol{n}-\mathbf{D}_1)+\boldsymbol{d}}-\frac{\mathbf{E}}{\boldsymbol{k}(\mathbf{E}_1\boldsymbol{n}-\mathbf{E}_1)+\boldsymbol{e}}\right]
$$

定理16 设A、B、D、b、d、e都是正整数，则

$$
\sum_{n=1}^{\infty}\left[\frac{A}{k(2n-2)+a}-\frac{B}{k(3n-3)+b}+\frac{2D}{k(4n-4)+d}\right.
$$

$$
\left.-\frac{3A+3D-2B}{k(6n-6)+e}\right]
$$

$$
=AC(a,2k)-BC(b,3k)+2DC(d,4k)
$$

$$
-(3A+3D-2B)C(e,6k)
$$

$$
+\frac{1}{6k}\ln\frac{2^{3A}4^{3D}b^{2B}e^{3A+3D-2B}}{3^{2B}6^{3A+3D-2B}a^{3A}d^{3D}};\qquad(1)
$$

$$
\sum_{n=1}^{\infty}\left[\frac{3A}{k(3n-3)+a}-\frac{B}{k(4n-4)+b}+\frac{3D}{k(6n-6)+d}\right.
$$

$$
\left.-\frac{8A+4D-2B}{k(8n-8)+e}\right]
$$

$$
=3AC(a,3k)-BC(b,4k)+3DC(d,6k)
$$

$$
-(8A+4D-2B)C(e,8k)
$$

$$+\frac{1}{4k}\ln\frac{3^{4A}6^{2D}b^{B}e^{4A+2D-B}}{4^{B}8^{4A+2D-B}a^{4A}d^{2D}};\tag{2}$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(n-1)+a}-\frac{B}{k(2n-2)+b}+\frac{3D}{k(3n-3)+d}\right.$$

$$\left.-\frac{4A+4D-2B}{k(4n-4)+e}\right]$$

$$=AC(a,k)-BC(b,2k)+3DC(d,3k)$$

$$-(4A+4D-2B)C(e,4k)$$

$$+\frac{1}{2k}\ln\frac{3^{2D}b^{B}e^{2A+2D-B}}{2^{B}4^{2A+2D-B}a^{2A}d^{2D}};\tag{3}$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(2n-2)+a}-\frac{3B}{k(3n-3)+b}+\frac{3D}{k(6n-6)+d}\right.$$

$$\left.-\frac{4A+4D-8B}{k(8n-8)+e}\right]$$

$$=AC(a,2k)-3BC(b,3k)+3DC(d,6k)$$

$$-(4A+4D-8B)C(e,8k)$$

$$+\frac{1}{2k}\ln\frac{2^{A}6^{D}b^{2B}e^{A+D-2B}}{3^{2B}8^{A+D-2B}a^{A}d^{D}};\tag{4}$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(3n-3)+a}-\frac{B}{k(6n-6)+b}+\frac{2D}{k(8n-8)+d}\right.$$

$$\left.-\frac{4A+3B-2B}{k(12n-12)+e}\right]$$

$$=AC(a,3k)-BC(b,6k)+2DC(d,8k)$$

$$-(4A+3D-2B)C(e,12k)$$

$$+\frac{1}{12k}\ln\frac{3^{4A}8^{3D}b^{2B}e^{4A+3D-2B}}{6^{2B}12^{4A+3D-2B}a^{4A}d^{3D}};\tag{5}$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(2n-2)+a}-\frac{3B}{k(3n-3)+b}+\frac{D}{k(4n-4)+d}\right.$$

$$- \frac{4A+2D-8B}{k(8n-8)+e}\Bigg]$$

$$= AC(a,2k) - 3BC(b,3k) + DC(d,4k)$$

$$-(4A+2D-8B)C(e,8k)$$

$$+ \frac{1}{4k}\ln\frac{2^{2A}4^{D}b^{4B}e^{2A+D-4B}}{3^{4B}8^{2A+D-4B}a^{2A}d^{D}}; \qquad (6)$$

$$\sum_{n=1}^{\infty}\Bigg[\frac{A}{k(3n-3)+a} - \frac{B}{k(4n-4)+b} + \frac{2D}{k(8n-8)+d}$$

$$- \frac{4A+3D-3B}{k(12n-12)+e}\Bigg]$$

$$= AC(a,3k) - BC(b,4k) + 2DC(d,8k)$$

$$-(4A+3D-3B)C(e,12k)$$

$$+ \frac{1}{12k}\ln\frac{3^{4A}8^{3D}b^{3B}e^{4A+3D-3B}}{4^{3B}12^{4A+3D-3B}a^{4A}d^{3D}}; \qquad (7)$$

$$\sum_{n=1}^{\infty}\Bigg[\frac{A}{k(3n-3)+a} - \frac{B}{k(4n-4)+b} + \frac{D}{k(6n-6)+d}$$

$$- \frac{4A+2D-3B}{k(12n-12)+e}\Bigg]$$

$$= AC(a,3k) - BC(b,4k) + DC(d,6k)$$

$$-(4A+2D-3B)C(e,12k)$$

$$+ \frac{1}{12k}\ln\frac{3^{4A}6^{2D}b^{3B}e^{4A+2D-3B}}{4^{3B}12^{4A+2D-3B}a^{4A}d^{2D}}; \qquad (8)$$

$$\sum_{n=1}^{\infty}\Bigg[\frac{A}{k(3n-3)+a} - \frac{2B}{k(2n-2)+b} + \frac{2D}{k(2n-2)+d}$$

$$- \frac{A+3D-3B}{k(3n-3)+e}\Bigg]$$

$$= AC(a,3k) - 2BC(b,2k) + 2DC(d,2k)$$

$$-(A+3D-3B)C(e,3k)$$

$$+\frac{1}{3k}\ln\frac{3^{A}2^{3D}b^{3B}e^{A+3D-3B}}{2^{3B}3^{A+3D-3B}a^{A}d^{3D}}; \tag{9}$$

$$\sum_{n=1}^{\infty}\left[\frac{A}{k(4n-4)+a}+\frac{3B}{k(3n-3)+b}+\frac{3D}{k(3n-3)+d}\right.$$

$$\left.-\frac{A+4D-4B}{k(4n-4)+e}\right]$$

$$=AC(a,4k)-3BC(b,3k)+3DC(d,3k)$$

$$-(A+4D-4B)C(e,4k)$$

$$+\frac{1}{4k}\ln\frac{4^{A}3^{4D}b^{4B}e^{A+4D-4B}}{3^{4B}4^{A+4D-4B}a^{A}d^{4D}}; \tag{10}$$

$$\sum_{n=1}^{\infty}\left[\frac{3A}{k(3n-3)+a}-\frac{B}{k(2n-2)+b}+\frac{3D}{k(3n-3)+d}\right.$$

$$\left.-\frac{4A+4D-2B}{k(4n-4)+e}\right]$$

$$=3AC(a,3k)-BC(b,2k)+3DC(d,3k)$$

$$-(4A+4D-2B)C(e,4k)$$

$$+\frac{1}{2k}\ln\frac{3^{2A+2D}b^{B}e^{2A+2D-B}}{2^{4A+4D-B}a^{2A}d^{2D}}; \tag{11}$$

$$\sum_{n=1}^{\infty}\left[\frac{3A}{k(6n-6)+a}-\frac{B}{k(4n-4)+b}+\frac{3D}{k(3n-3)+d}\right.$$

$$\left.-\frac{2A+4D-B}{k(4n-4)+e}\right]$$

$$=3AC(a,6k)-BC(b,4k)+3DC(d,3k)$$

$$-(2A+4D-B)C(e,4k)$$

$$+\frac{1}{4k}\ln\frac{6^{2A}3^{4D}b^{B}e^{2A+4D-B}}{4^{2A+4D}a^{2A}d^{4D}}. \tag{12}$$

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{A}{k(2i-2)+a}-\frac{B}{k(3i-3)+d}+\frac{3D}{k(4i-4)+b}\right.$$
$$\left.-\frac{3A+3D-2B}{k(6i-6)+e}\right]$$
$$=AC(a,2k)-BC(b,3k)+2DC(d,4k)$$
$$-(3A+3D-2B)C(e,6k)$$
$$+\frac{1}{12k}\ \ln\frac{(2nk-2k+a)^{6A}(4nk-4k+d)^{6D}}{(3nk-3k+b)^{4B}(6nk-6k+e)^{6A+6D-4B}}$$
$$+\frac{1}{12k}\ \ln\frac{b^{4B}e^{6A+6D-4B}}{a^{6A}d^{6D}}+A\varepsilon_n(a,2k)-B\varepsilon_n(b,3k)$$
$$+2D\varepsilon_n(d,4k)-(3A+3D-2B)\varepsilon_n(e,6k).$$

令 $n\to\infty$ 并取极限，即得(1)。

仿此可证定理的其余各部分。

例1 求和：

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}.$$

解 用分项分式方法求得

$$\frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}$$
$$=\frac{1}{1540}\left(\frac{165}{6n-1}-\frac{154}{4n-1}+\frac{30}{4n-3}+\frac{21}{6n+1}\right).$$

于定理16(9)令 $A=165$，$B=77$，$D=15$，$a=5$，$b=3$，$d=1$，$e=7$，$k=2$，则有

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}$$
$$=\frac{1}{1540}\sum_{n=1}^{\infty}\left(\frac{165}{6n-1}-\frac{154}{4n-1}+\frac{30}{4n-3}+\frac{21}{6n+1}\right)$$

$$=\frac{3}{28}C(5,6)-\frac{1}{10}C(3,4)+\frac{3}{154}C(1,4)+\frac{3}{220}C(7,6)$$

$$+\frac{1}{9240}\ln\frac{3^{417}}{2^{186}5^{165}7^{21}}.$$

于命题2令 $a=1$，$b=1$，$k=6$，则有

$$C(7,6)=C(1,6)+\frac{1}{6}\ln 7-1,$$

$$\frac{3}{220}C(7,6)=\frac{3}{220}C(1,6)+\frac{1}{1320}\ln 7^3-\frac{3}{220}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}$$

$$=\frac{3}{28}C(5,6)-\frac{1}{10}C(3,4)+\frac{3}{154}C(1,4)+\frac{3}{220}C(1,6)$$

$$+\frac{1}{9240}\ln\frac{3^{417}}{2^{186}5^{165}}-\frac{3}{220}.$$

根据定理6，有

$$\frac{3}{28}C(5,6)-\frac{1}{14}C(3,4)$$

$$=-\frac{\pi\sqrt{3}}{112}+\frac{\pi}{112}+\frac{1}{112}\ln\frac{5^2}{3},$$

$$\frac{3}{154}C(1,4)-\frac{3}{154}C(3,4)$$

$$=\frac{3\pi}{616}-\frac{1}{616}\ln 3^3,$$

$$\frac{3}{220}C(1,6)-\frac{1}{110}C(3,4)$$

$$=\frac{\pi\sqrt{3}}{880}+\frac{\pi}{880}-\frac{1}{880}\ln 3.$$

于是 $\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}$

$$=-\frac{3\pi\sqrt{3}}{385}+\frac{23\pi}{1540}+\frac{93}{3080}\ln 3-\frac{31}{1540}\ln 2$$

$$-\frac{3}{220}.$$

例2 求和:

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)(8n+1)}$$

解 用分项分式方法求得

$$\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)}$$

$$=\frac{1}{20}\left(\frac{5}{2n-1}-\frac{20}{3n-1}+\frac{16}{4n-1}+\frac{1}{6n+1}\right)$$

于定理16(1) 令 $A=5$, $B=20$, $D=8$, $a=1$, $b=2$, $d=3$, $e=7$, $k=1$, 则有

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)}$$

$$=\frac{1}{20}\sum_{n=1}^{\infty}\left(\frac{5}{2n-1}-\frac{20}{3n-1}+\frac{16}{4n-1}+\frac{1}{6n+1}\right)$$

$$=\frac{1}{4}C(1,2)-C(2,3)+\frac{4}{5}C(3,4)+\frac{1}{20}C(7,6)$$

$$+\frac{1}{120}\ln\frac{2^{104}}{3^{63}\cdot 7}.$$

于命题2令 $a=1$, $b=1$, $k=6$, 则有

$$C(7,6)=C(1,6)+\frac{1}{6}\ln 7-1,$$

$$\frac{1}{20}C(7,6)=\frac{1}{20}C(1,6)+\frac{1}{120}\ln 7-\frac{1}{20}.$$

于是 $\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)}$

$$=\frac{1}{4}C(1,2)-C(2,3)+\frac{4}{5}C(3,4)+\frac{1}{20}C(1,6)$$

$$+\frac{1}{120}\ln\frac{2^{104}}{3^{63}}+\frac{1}{20}.$$

根据定理6，有

$$\frac{1}{4}C(1,2)-\frac{3}{8}C(2,3)=\frac{\pi}{16\sqrt{3}}-\frac{1}{16}\ln 3,$$

$$\frac{4}{5}C(3,4)-\frac{3}{5}C(2,3)=\frac{\pi}{10\sqrt{3}}-\frac{\pi}{10}+\frac{1}{10}\ln 3,$$

$$\frac{1}{20}C(1,6)-\frac{1}{40}C(2,3)=\frac{\pi}{60\sqrt{3}}.$$

于是 $\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)}$

$$=\frac{43\pi}{240\sqrt{3}}-\frac{\pi}{10}+\frac{13}{15}\ln 2-\frac{39}{80}\ln 3-\frac{1}{20}.$$

用分项分式方法求得

$$\frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$=\frac{1}{165}\left(\frac{15}{3n-1}-\frac{22}{4n-1}+\frac{33}{6n+1}-\frac{40}{8n+1}\right).$$

于定理16（2）令 $A=5$，$B=22$，$D=11$，$a=2$，$b=3$，$d=7$，$e=9$，$k=1$，则有

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$=\frac{1}{165}\sum_{n=1}^{\infty}\left(\frac{15}{3n-1}-\frac{22}{4n-1}+\frac{33}{6n+1}-\frac{40}{8n+1}\right)$$

$$=\frac{1}{11}C(2,3)-\frac{2}{15}C(3,4)+\frac{1}{5}C(7,6)-\frac{8}{33}C(9,8)$$

$$+\frac{1}{660}\ln\frac{3^{104}}{2^{102}7^{22}}.$$

于命题2令 $a=1$，$b=1$，$k=6$，则有

$$C(7,6)=C(1,6)+\frac{1}{6}\ln 7-1,$$

$$\frac{1}{5}C(7,6)=\frac{1}{5}C(1,6)+\frac{1}{30}\ln 7-\frac{1}{5}.$$

于命题2令 $a=1$，$b=1$，$k=8$，则有

$$C(9,8)=C(1,8)+\frac{1}{8}\ln 3^2-1,$$

$$-\frac{8}{33}C(9,8)=-\frac{8}{33}C(1,8)-\frac{1}{33}\ln 3^2+\frac{8}{33}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$=\frac{1}{11}C(2,3)-\frac{2}{15}C(3,4)+\frac{1}{5}C(1,6)-\frac{8}{33}C(1,8)$$

$$+\frac{1}{660}\ln\frac{3^{64}}{2^{102}}+\frac{7}{165}.$$

根据定理6，有

$$\frac{1}{11}C(2,3)-\frac{4}{33}C(3,4)=-\frac{\pi}{66\sqrt{3}}+\frac{\pi}{66}-\frac{1}{66}\ln 3,$$

$$\frac{1}{55}C(1,6)-\frac{2}{165}C(3,4)$$

$$=\frac{\pi}{220\sqrt{3}}+\frac{\pi}{660}-\frac{1}{660}\ln 3,$$

$$\frac{2}{11}C(1,6)-\frac{8}{33}C(1,8)$$

$$=\frac{\pi}{22\sqrt{3}}-\frac{\pi}{33\sqrt{2}}-\frac{\pi}{66}+\frac{1}{66}\ln 3-\frac{\sqrt{2}}{33}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$=\frac{23\pi}{660\sqrt{3}}-\frac{\pi}{33\sqrt{2}}+\frac{\pi}{660}+\frac{21}{220}\ln 3-\frac{17}{110}\ln 2$$

$$-\frac{\sqrt{2}}{33}\ln(\sqrt{2}+1)+\frac{7}{165}.$$

所以 $$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)}\right.$$

$$\left.-\frac{4n}{(3n-1)(4n-1)(6n+1)(8n+1)}\right]$$

$$=\frac{7\pi}{880\sqrt{3}}+\frac{2\pi\sqrt{2}}{165}-\frac{7\pi}{330}+\frac{49}{165}\ln 2-\frac{153}{880}\ln 3$$

$$+\frac{4\sqrt{2}}{165}\ln(\sqrt{2}+1)-\frac{29}{660}.$$

例3 求和：

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 用分项分式方法求得

$$\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\frac{1}{4}\left(\frac{1}{2n-1}-\frac{4}{3n-1}+\frac{4}{4n-1}-\frac{1}{6n-1}\right).$$

于定理16(1)令 $A=1$, $B=4$, $D=2$, $a=1$, $b=2$, $d=3$, $e=5$, $k=1$, 则有

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{4}{3n-1}+\frac{4}{4n-1}-\frac{1}{6n-1}\right)$$

$$=\frac{1}{4}C(1,2)-C(2,3)+C(3,4)-\frac{1}{4}C(5,6)$$

$$+\frac{1}{24}\ln\frac{2^{22}5}{3^{15}}.$$

根据定理6，有

$$\frac{1}{4}C(1,2)-\frac{3}{8}C(2,3)=\frac{\pi}{16\sqrt{3}}-\frac{1}{16}\ln 3,$$

$$\frac{5}{6}C(3,4)-\frac{5}{8}C(2,3)=\frac{5\pi}{48\sqrt{3}}-\frac{5\pi}{48}+\frac{1}{48}\ln 3^5,$$

$$\frac{1}{6}C(3,4)-\frac{1}{4}C(5,6)=\frac{\pi}{16\sqrt{3}}-\frac{\pi}{48}+\frac{1}{48}\ln\frac{3}{5^2}.$$

于是 $$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\frac{11\pi}{48\sqrt{3}}-\frac{\pi}{8}+\frac{11}{12}\ln 2-\frac{9}{16}\ln 3.$$

于定理16(2)令 $A=2$，$B=15$，$D=5$，$a=2$，$b=3$，$d=5$，$e=7$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{30}\sum_{n=1}^{\infty}\left(\frac{6}{3n-1}-\frac{15}{4n-1}+\frac{15}{6n-1}-\frac{6}{8n-1}\right)$$

$$=-\frac{19\pi}{120\sqrt{3}}+\frac{\pi}{40\sqrt{2}}+\frac{3\pi}{40}+\frac{9}{40}\ln 3-\frac{37}{120}\ln 2$$

$$-\frac{1}{20\sqrt{2}}\ln(\sqrt{2}+1).$$

于定理16(1)令 $A=1$，$B=4$，$D=2$，$a=3$，$b=5$，$d=7$，$e=11$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{4}{6n-1}+\frac{4}{8n-1}-\frac{1}{12n-1}\right)$$

$$=\frac{3\pi\sqrt{3}}{128}-\frac{\pi}{32\sqrt{2}}-\frac{7\pi}{384}+\frac{7}{96}\ln 2-\frac{9}{128}\ln 3$$

$$+\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{64\sqrt{3}}\ln(2+\sqrt{3}).$$

由此顺次推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{4n^2}{(3n-1)(4n-1)(6n-1)(8n-1)}\right]$$

$$=\frac{23\pi}{80\sqrt{3}}-\frac{\pi}{30\sqrt{2}}-\frac{17\pi}{120}+\frac{43}{60}\ln 2-\frac{39}{80}\ln 3$$

$$+\frac{1}{15\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{4n^2}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=-\frac{211\pi}{1440\sqrt{3}}+\frac{\pi}{20\sqrt{2}}+\frac{71\pi}{1440}+\frac{27}{160}\ln 3-\frac{1}{5}\ln 2$$

$$+\frac{1}{48\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1),$$

于是

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6n^2}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{7\pi}{30\sqrt{3}}-\frac{\pi}{15\sqrt{2}}-\frac{7\pi}{80}+\frac{23}{60}\ln 2-\frac{3}{10}\ln 3$$

$$+\frac{\sqrt{2}}{15}\ln(\sqrt{2}+1)-\frac{1}{40\sqrt{3}}\ln(2+\sqrt{3}).$$

例4 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}.$$

解 于定理16(1)令$A=9$，$B=64$，$D=50$，$a=1$，$b=2$，$d=3$，$e=5$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\left(\frac{9}{4n-3}-\frac{64}{6n-4}+\frac{100}{8n-5}-\frac{49}{12n-7}\right)$$

$$=\frac{19\pi}{384\sqrt{3}}+\frac{25\pi}{32\sqrt{2}}-\frac{221\pi}{384}+\frac{89}{32}\ln 2-\frac{177}{128}\ln 3$$

$$+\frac{49}{64\sqrt{3}}\ln(2+\sqrt{3})-\frac{25}{16\sqrt{2}}\ln(\sqrt{2}+1).$$

由此以及上面已求出的相应的结果即可推得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(4n-3)(6n-4)(8n-5)(12n-7)}\right.$$

$$\left.-\frac{4n^2}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=4\sum_{n=1}^{\infty}\frac{n^2}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$-4\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$+\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)}$$

$$-4\sum_{n=1}^{\infty}\frac{n^2}{(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=-\frac{\pi}{3\sqrt{3}}+\frac{\pi}{4\sqrt{2}}+\frac{\pi}{48}+\frac{1}{3}\ln 2$$

$$+\frac{1}{8\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1).$$

例5 求和：

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)}.$$

解 于定理 16 (11) 令 $A=5$，$B=27$，$D=16$，$a=1$，$b=1$，$d=2$，$e=3$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n-3)(6n-5)(6n-4)(8n-5)}$$

$$=\frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{15}{6n-5}-\frac{27}{4n-3}+\frac{48}{6n-4}-\frac{30}{8n-5}\right)$$

$$=\frac{31\pi}{48\sqrt{3}}-\frac{5\pi}{16\sqrt{2}}-\frac{\pi}{8}+\frac{21}{16}\ln 3-\frac{121}{48}\ln 2$$

$$+\frac{5}{8\sqrt{2}}\ln(\sqrt{2}+1).$$

于定理16(11)，令 $A=1$，$B=6$，$D=4$, $a=1$, $b=1$, $d=2$, $e=3$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(6n-5)(6n-4)(8n-5)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{6n-5}-\frac{6}{4n-3}+\frac{12}{6n-4}-\frac{8}{8n-5}\right)$$

$$=\frac{7\pi}{8\sqrt{3}}-\frac{\pi}{2\sqrt{2}}-\frac{\pi}{8}+\frac{15}{8}\ln 3-\frac{15}{4}\ln 2$$

$$+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

于 定 理 16 (11) 令 $A=3$，$B=20$，$D=15$，$a=1$，$b=1$，$d=2$，$e=3$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{9}{6n-5}-\frac{20}{4n-3}+\frac{45}{6n-4}-\frac{32}{8n-5}\right)$$

$$=\frac{2\pi\sqrt{3}}{5}-\frac{2\pi\sqrt{2}}{5}-\frac{\pi}{10}+\frac{27}{10}\ln 3-\frac{28}{5}\ln 2$$

$$+\frac{4\sqrt{2}}{5}\ln(\sqrt{2}+1).$$

根据例 3 中的结果，有

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n-1)(6n-2)(6n-1)(8n-1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$=-\frac{19\pi}{240\sqrt{3}}+\frac{\pi}{80\sqrt{2}}+\frac{3\pi}{80}+\frac{9}{80}\ln 3-\frac{37}{240}\ln 2$$

$$-\frac{1}{40\sqrt{2}}\ln(\sqrt{2}+1).$$

于是 $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(4n-3)(6n-5)(6n-4)(8n-5)}\right.$$

$$\left.-\frac{4n^2}{(4n-1)(6n-2)(6n-1)(8n-1)}\right]$$

$$=\frac{\pi\sqrt{3}}{5}-\frac{\pi}{10\sqrt{2}}-\frac{\pi}{4}-\frac{1}{15}\ln 2$$

$$+\frac{1}{5\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(4n-3)(6n-5)(6n-4)(8n-5)}\right.$$

$$\left.+\frac{4n^2}{(4n-1)(6n-2)(6n-1)(8n-1)}\right]$$

$$=-\frac{\pi}{30\sqrt{3}}+\frac{\pi}{20}+\frac{9}{10}\ln 3-\frac{13}{10}\ln 2.$$

例6 求和:

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理16(3)令 $A=1, B=6, D=3, a=1, b=3, d=5, e=7, k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{24}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{6}{4n-1}+\frac{9}{6n-1}-\frac{4}{8n-1}\right)$$

$$= -\frac{\pi\sqrt{3}}{32} + \frac{\pi}{48\sqrt{2}} + \frac{\pi}{24} + \frac{3}{32}\ln 3 - \frac{5}{48}\ln 2$$

$$-\frac{1}{24\sqrt{2}}\ln(\sqrt{2}+1).$$

由此以及例3中的相应结果即可求得

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6n^2}{(4n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$= -\frac{11\pi\sqrt{3}}{320} + \frac{\pi}{24\sqrt{2}} + \frac{29\pi}{960} + \frac{33}{320}\ln 3 - \frac{13}{120}\ln 2$$

$$+\frac{\sqrt{3}}{160}\ln(2+\sqrt{3}) - \frac{1}{12\sqrt{2}}\ln(\sqrt{2}+1).$$

例7 求和:

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理16(4)令 $A=5, B=4, D=5, a=1, b=2, d=5, e=7, k=1$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)}$$

$$=\frac{1}{60}\sum_{n=1}^{\infty}\left(\frac{5}{2n-1} - \frac{12}{3n-1} + \frac{15}{6n-1} - \frac{8}{8n-1}\right)$$

$$= -\frac{7\pi}{240\sqrt{3}} + \frac{\pi}{60\sqrt{2}} + \frac{\pi}{120} + \frac{1}{10}\ln 2 - \frac{3}{80}\ln 3$$

$$-\frac{1}{30\sqrt{2}}\ln(\sqrt{2}+1).$$

于定理16(5)令 $A=2$, $B=15$, $D=9, a=2, b=5, d=7$,

$e=11$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{90}\sum_{n=1}^{\infty}\left(\frac{2}{3n-1}-\frac{15}{6n-1}+\frac{18}{8n-1}-\frac{5}{12n-1}\right)$$

$$=\frac{97\pi}{2160\sqrt{3}}-\frac{\pi}{40\sqrt{2}}-\frac{17\pi}{2160}+\frac{11}{360}\ln 2-\frac{3}{80}\ln 3$$

$$+\frac{1}{20\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{72\sqrt{3}}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)}\right.$$

$$\left.-\frac{6n^2}{(3n-1)(6n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{97\pi}{2160\sqrt{3}}-\frac{\pi}{40\sqrt{2}}-\frac{17\pi}{2160}+\frac{11}{360}\ln 2-\frac{3}{80}\ln 3$$

$$+\frac{1}{20\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{72\sqrt{3}}\ln(2+\sqrt{3}).$$

例8 求和：

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}.$$

解 于定理16（6）令 $A=5$，$B=6$，$D=15$，$a=1$，$b=2$，$d=3$，$e=7$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)}$$

$$=\frac{1}{30}\sum_{n=1}^{\infty}\left(\frac{5}{2n-1}-\frac{18}{3n-1}+\frac{15}{4n-1}-\frac{2}{8n-1}\right)$$

$$=\frac{\pi}{10\sqrt{3}}+\frac{\pi}{120\sqrt{2}}-\frac{7\pi}{120}+\frac{61}{120}\ln 2-\frac{3}{10}\ln 3$$

$$-\frac{1}{60\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理16(7) 令 $A=8$, $B=15$, $D=6$, $a=2$, $b=3$, $d=7$, $e=11$, $k=1$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{120}\sum_{n=1}^{\infty}\left(\frac{8}{3n-1}-\frac{15}{4n-1}+\frac{12}{8n-1}-\frac{5}{12n-1}\right)$$

$$=-\frac{17\pi}{2880\sqrt{3}}-\frac{\pi}{80\sqrt{2}}+\frac{37\pi}{2880}+\frac{9}{320}\ln 3-\frac{13}{240}\ln 2$$

$$+\frac{1}{40\sqrt{2}}\ln(\sqrt{2}+1)-\frac{1}{96\sqrt{3}}\ln(2+\sqrt{3}).$$

于是 $$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)}\right.$$

$$\left.-\frac{6n^2}{(3n-1)(4n-1)(8n-1)(12n-1)}\right]$$

$$=\frac{13\pi}{480\sqrt{3}}+\frac{\pi}{60\sqrt{2}}-\frac{13\pi}{480}+\frac{1}{6}\ln 2-\frac{3}{32}\ln 3$$

$$+\frac{1}{80\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{30\sqrt{2}}\ln(\sqrt{2}+1).$$

例9 求和:

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}.$$

解 于定理16(8)令 $A=4$, $B=9$, $D=6$, $a=2$, $b=3$, $d=5$, $e=11$, $k=1$, 即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{36}\sum_{n=1}^{\infty}\left(\frac{4}{3n-1}-\frac{9}{4n-1}+\frac{6}{6n-1}-\frac{1}{12n-1}\right)$$

$$=-\frac{49\pi}{864\sqrt{3}}+\frac{29\pi}{864}+\frac{3}{32}\ln 3-\frac{5}{36}\ln 2$$

$$-\frac{1}{144\sqrt{3}}\ln(2+\sqrt{3}).$$

由此以及例 3 中的相应结果即可求得

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$=\frac{1}{5}\sum_{n=1}^{\infty}\left[\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{6n^2}{(3n-1)(4n-1)(6n-1)(12n-1)}\right]$$

$$=\frac{41\pi}{360\sqrt{3}}-\frac{47\pi}{720}+\frac{7}{20}\ln 2-\frac{9}{40}\ln 3$$

$$+\frac{1}{120\sqrt{3}}\ln(2+\sqrt{3}).$$

仿此求得以下一系列的结果：

(一)于定理16(2)令 $A=33$, $B=154$, $D=21$, $a=2$, $b=3$, $d=5$, $e=3$, $k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)(8n-5)}$$

$$=-\frac{1}{231}\sum_{n=1}^{\infty}\left(\frac{99}{3n-1}-\frac{154}{4n-1}+\frac{63}{6n-1}-\frac{40}{8n-5}\right)$$

$$=\frac{43\pi}{308\sqrt{3}}+\frac{5\pi}{231\sqrt{2}}-\frac{29\pi}{308}+\frac{229}{462}\ln 2$$

$$-\frac{87}{308}\ln 3-\frac{5\sqrt{2}}{231}\ln(\sqrt{2}+1);$$

于定理16(1)，令 $A=55$，$B=36$，$D=200$，$a=3$，$b=5$，$d=3$，$e=5$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n}{(4n-1)(6n-1)(8n-5)(12n-7)}$$

$$=\frac{1}{1320}\sum_{n=1}^{\infty}\left(\frac{55}{4n-1}-\frac{36}{6n-1}+\frac{400}{8n-5}-\frac{693}{12n-7}\right)$$

$$=\frac{17\pi\sqrt{3}}{704}+\frac{5\pi}{132\sqrt{2}}-\frac{239\pi}{3520}+\frac{7}{165}\ln 2-\frac{51}{704}\ln 3$$

$$-\frac{5}{66\sqrt{2}}\ln(\sqrt{2}+1)+\frac{7\sqrt{3}}{160}\ln(2+\sqrt{3}).$$

由此以及上面已求出的相应的结果，即可顺次求得

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-5)}$$

$$=-\sum_{n=1}^{\infty}\left[\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}\right.$$

$$\left.-\frac{4n}{(3n-1)(4n-1)(6n-1)(8n-5)}\right]$$

$$=-\frac{195\pi}{616\sqrt{3}}+\frac{10\pi\sqrt{2}}{231}+\frac{19\pi}{154}-\frac{235}{231}\ln 2$$

$$+\frac{459}{616}\ln 3-\frac{20\sqrt{2}}{231}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)(8n-5)(12n-7)}$$

$$=-\frac{1}{3}\sum_{n=1}^{\infty}\frac{n}{(3n-1)(4n-1)(6n-1)(8n-5)}$$

$$-\frac{4n}{(4n-1)(6n-1)(8n-5)(12n-7)}\Big]$$

$$=\frac{185\pi}{3696\sqrt{3}}+\frac{10\pi}{231\sqrt{2}}-\frac{1093\pi}{18480}-\frac{251}{2310}\ln 2-\frac{3}{1232}\ln 3$$

$$-\frac{10\sqrt{2}}{231}\ln(\sqrt{2}+1)+\frac{7}{40\sqrt{3}}\ln(2+\sqrt{3}).$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-5)(12n-7)}$$

$$=-\sum_{n=1}^{\infty}\Big[\frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-5)}$$

$$-\frac{6n}{(3n-1)(4n-1)(6n-1)(8n-5)(12n-7)}\Big]$$

$$=\frac{95\pi}{154\sqrt{3}}+\frac{20\pi\sqrt{2}}{231}-\frac{1473\pi}{3080}+\frac{422}{1155}\ln 2-\frac{117}{154}\ln 3$$

$$-\frac{40\sqrt{2}}{231}\ln(\sqrt{2}+1)+\frac{7\sqrt{3}}{20}\ln(2+\sqrt{3}).$$

(二)于定理16(10)，令 $A=147$，$B=50$，$D=32,a=1$，$b=1$，$d=2$，$e=3$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)}$$

$$=\frac{1}{120}\sum_{n=1}^{\infty}\left(\frac{147}{8n-7}-\frac{150}{6n-5}+\frac{96}{6n-4}-\frac{75}{8n-5}\right)$$

$$=-\frac{59\pi}{240\sqrt{3}}+\frac{3\pi}{40\sqrt{2}}+\frac{37\pi}{320}-\frac{9}{80}\ln 3-\frac{7}{60}\ln 2$$

$$+\frac{37}{80\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理16(12)，令 $A=1210$，$B=3087$，$D=448$，$a=1$，$b=1$，$d=2$，$e=3$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$=\frac{1}{5040}\sum_{n=1}^{\infty}\left(\frac{3630}{12n-11}-\frac{3087}{8n-7}+\frac{1344}{6n-4}-\frac{1125}{8n-5}\right)$$

$$=\frac{2263\pi}{20160\sqrt{3}}-\frac{117\pi}{1120\sqrt{2}}+\frac{1439\pi}{40320}+\frac{351}{2240}\ln 3$$

$$-\frac{799}{3360}\ln 2+\frac{121}{672\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{109}{1120\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理16(9)，令 $A=121$，$B=49$，$D=25$，$a=1$，$b=1$，$d=3$，$e=5$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{448}\sum_{n=1}^{\infty}\left(\frac{121}{12n-11}-\frac{98}{8n-7}+\frac{50}{8n-5}-\frac{49}{12n-7}\right)$$

$$=\frac{85\pi}{1792\sqrt{3}}-\frac{3\pi}{224\sqrt{2}}-\frac{13\pi}{1792}+\frac{9}{448}\ln 3-\frac{3}{224}\ln 2$$

$$+\frac{85}{896\sqrt{3}}\ln(2+\sqrt{3})-\frac{37}{448\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理16(10)，令 $A=27$，$B=8$，$D=2$，$a=5$，$b=4$，$d=5$，$e=7$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)}$$

$$=\frac{1}{120}\sum_{n=1}^{\infty}\left(\frac{27}{8n-3}-\frac{24}{6n-2}+\frac{6}{6n-1}-\frac{3}{8n-1}\right)$$

$$=\frac{\pi}{240\sqrt{3}}-\frac{\pi}{40\sqrt{2}}+\frac{\pi}{64}+\frac{7}{60}\ln 2-\frac{3}{80}\ln 3$$

$$-\frac{1}{16\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理16(12)，令 $A=250$，$B=567$，$D=28$，$a=7$，$b=5$，

$d=5$，$e=7$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$=\frac{1}{5040}\sum_{n=1}^{\infty}\left(\frac{750}{12n-5}-\frac{567}{8n-3}+\frac{84}{6n-1}-\frac{45}{8n-1}\right)$$

$$=\frac{97\pi}{6720\sqrt{3}}+\frac{17\pi}{1120\sqrt{2}}-\frac{761\pi}{40320}+\frac{51}{2240}\ln 3$$

$$-\frac{181}{10080}\ln 2-\frac{25}{672\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{29}{1120\sqrt{2}}\ln(\sqrt{2}+1);$$

于定理16（9），令 $A=25$，$B=9$，$D=1$，$a=7$，$b=5$，$d=7$，$e=11$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\frac{n^2}{(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{448}\sum_{n=1}^{\infty}\left(\frac{25}{12n-5}-\frac{18}{8n-3}+\frac{2}{8n-1}-\frac{1}{12n-1}\right)$$

$$=\frac{13\pi}{1792\sqrt{3}}+\frac{\pi}{224\sqrt{2}}-\frac{13\pi}{1792}+\frac{3}{448}\ln 3-\frac{1}{224}\ln 2$$

$$+\frac{5}{448\sqrt{2}}\ln(\sqrt{2}+1)-\frac{13}{896\sqrt{3}}\ln(2+\sqrt{3}).$$

由此以及上面已求出的相应结果，可顺次求得

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)}\right]$$

$$= -\frac{\pi}{5\sqrt{3}} + \frac{\pi}{5\sqrt{2}} - \frac{1}{3}\ln 2 + \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left. + \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)}\right]$$

$$= -\frac{\pi}{6\sqrt{3}} + \frac{\pi}{8} + \frac{3}{5}\ln 2 - \frac{3}{10}\ln 3;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left. - \frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$= \frac{\pi}{45\sqrt{3}} - \frac{17\pi}{140\sqrt{2}} + \frac{25\pi}{252} - \frac{1}{45}\ln 2$$

$$+ \frac{25}{84\sqrt{3}}\ln(2+\sqrt{3}) - \frac{29}{140\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left. + \frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$= \frac{347\pi}{2520\sqrt{3}} - \frac{29\pi}{560} + \frac{51}{280}\ln 3 - \frac{209}{1260}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{4n^2}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$= -\frac{\pi}{28\sqrt{2}}+\frac{\pi}{28}+\frac{13}{112\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{5}{56\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{4n^2}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$= \frac{13\pi}{224\sqrt{3}}-\frac{5\pi}{224}+\frac{3}{56}\ln 3-\frac{1}{28}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$= -\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)}\right.$$

$$\left.-\frac{2n^2}{(6n-4)(8n-7)(8n-5)(12n-11)}\right]$$

$$= \frac{4741\pi}{10080\sqrt{3}}-\frac{159\pi}{560\sqrt{2}}-\frac{223\pi}{5040}+\frac{477}{1120}\ln 3$$

$$-\frac{201}{560}\ln 2+\frac{121}{336\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{23}{35\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{2n^2}{(8n-7)(8n-5)(12n-11)(12n-7)}\right]$$

$$=\frac{701\pi}{40320\sqrt{3}}-\frac{87\pi}{1120\sqrt{2}}+\frac{253\pi}{5040}+\frac{261}{2240}\ln 3$$

$$-\frac{709}{3360}\ln 2-\frac{13}{1344\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{19}{280\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$=-\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)}\right.$$

$$\left.-\frac{2n^2}{(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$=\frac{83\pi}{3360\sqrt{3}}+\frac{31\pi}{560\sqrt{2}}-\frac{269\pi}{5040}+\frac{93}{1120}\ln 3$$

$$-\frac{769}{5040}\ln 2-\frac{25}{336\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{2\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-1)(8n-3)(8n-1)(12n-5)}\right.$$

$$\left.-\frac{2n^2}{(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$= -\frac{\pi}{13440\sqrt{3}} + \frac{\pi}{160\sqrt{2}} - \frac{11\pi}{2520} + \frac{3}{320}\ln 3$$

$$-\frac{13}{1440}\ln 2 - \frac{11}{1344\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{1}{280\sqrt{2}}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$= \frac{11\pi}{45\sqrt{3}} - \frac{31\pi}{70\sqrt{2}} + \frac{25\pi}{126} + \frac{13}{45}\ln 2$$

$$+\frac{25}{42\sqrt{3}}\ln(2+\sqrt{3}) - \frac{16\sqrt{2}}{35}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.+\frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right]$$

$$= \frac{557\pi}{1260\sqrt{3}} - \frac{8\pi}{35} + \frac{93}{140}\ln 3 - \frac{587}{630}\ln 2;$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.-\frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{\pi}{45\sqrt{3}}-\frac{\pi}{20\sqrt{2}}+\frac{\pi}{36}-\frac{1}{45}\ln 2$$

$$+\frac{11}{168\sqrt{3}}\ln(2+\sqrt{3})-\frac{1}{35\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\left[\frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right.$$

$$\left.+\frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\right]$$

$$=\frac{109\pi}{5040\sqrt{3}}-\frac{\pi}{140}+\frac{3}{40}\ln 3-\frac{119}{1260}\ln 2.$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}\right.$$

$$\left.-\frac{2n^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}\right]$$

$$=\frac{2927\pi}{20160\sqrt{3}}-\frac{3\pi}{70\sqrt{2}}-\frac{27\pi}{560}+\frac{9}{140}\ln 3$$

$$+\frac{53}{2520}\ln 2+\frac{85}{672\sqrt{3}}\ln(2+\sqrt{3})$$

$$-\frac{37}{140\sqrt{2}}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left[\frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}\right.$$

$$-\frac{2n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{167\pi}{20160\sqrt{3}}+\frac{\pi}{70\sqrt{2}}-\frac{5\pi}{336}+\frac{3}{140}\ln 3$$

$$-\frac{113}{2520}\ln 2-\frac{13}{672\sqrt{3}}\ln(2+\sqrt{3})$$

$$+\frac{1}{28\sqrt{2}}\ln(\sqrt{2}+1).$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$-\frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{\pi}{15\sqrt{3}}-\frac{2\pi\sqrt{2}}{35}+\frac{\pi}{21}+\frac{1}{9}\ln 2$$

$$+\frac{13}{84\sqrt{3}}\ln(2+\sqrt{3})-\frac{\sqrt{2}}{7}\ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$=\sum_{n=1}^{\infty}\Bigg[\frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$+\frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}\Bigg]$$

$$=\frac{67\pi}{504\sqrt{3}}-\frac{\pi}{14}+\frac{6}{35}\ln 3-\frac{26}{105}\ln 2.$$

四、一般的结果

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-2)+a}$$

$$\left.-\frac{m-1}{k(mn-1)+a}\right]$$

定理17 设m为正整数，且$m\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-2)+a}$$

$$\left.-\frac{m-1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-k,mk)$$

$$+\frac{1}{k}\ln\frac{a+mk-k}{a}.$$

证明 根据定理2，有

$$\sum_{i=1}^{n}\left[\frac{1}{k(mi-m)+a}+\frac{1}{k(mi-m+1)+a}\right.$$

$$+\frac{1}{k(mi-m+2)+a}+\cdots+\frac{1}{k(mi-2)+a}$$

$$\left.-\frac{m-1}{k(mi-1)+a}\right]$$

$$=C(a,mk)+C(a+k,mk)+C(a+2k,mk)+\cdots$$

$$+C(a+mk-2k,mk)-(m-1)C(a+mk-k,mk)$$

$$+\frac{1}{mk}\ \ln\frac{(mnk-mk+a)\cdots(mnk-2k+a)}{(mnk-k+a)^{m-1}}$$

$$-\frac{1}{mk}\ \ln\frac{a(a+k)(a+2k)\cdots(a+mk-2k)}{(a+mk-k)^{m-1}}$$

$$+\varepsilon_n(a,mk)+\varepsilon_n(a+k,mk)+\varepsilon_n(a+2k,mk)$$

$$+\cdots+\varepsilon_n(a+mk-2k,mk)$$

$$-(m-1)\varepsilon_n(a+mk-k,mk).$$

令 $n\to\infty$ 并取极限即得

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-2)+a}$$

$$\left.-\frac{m-1}{k(mn-1)+a}\right]$$

$$=C(a,mk)+C(a+k,mk)+C(a+2k,mk)+\cdots$$

$$+C(a+mk-2k,mk)-(m-1)C(a+mk-k,mk)$$

$$-\frac{1}{mk}\ \ln\frac{a(a+k)(a+2k)\cdots(a+mk-2k)}{(a+mk-k)^{m-1}}.$$

根据定理4，有

$$C(a,mk)+C(a+k,mk)+C(a+2k,mk)$$

$$+\cdots+C(a+mk-2k,mk)$$

$$=C(a,k)-C(a+mk-k,mk)$$
$$+\frac{1}{mk}\ln\frac{(a+k)(a+2k)\cdots(a+mk-k)}{a^{m-1}}.$$

于是 $\sum\limits_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$
$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-2)+a}$$
$$\left.-\frac{m-1}{k(mn-1)+a}\right]$$
$$=C(a,k)-mC(a+mk-k,mk)$$
$$+\frac{1}{mk}\ln\frac{(a+k)(a+2k)\cdots(a+mk-k)}{a^{m-1}}$$
$$-\frac{1}{mk}\ln\frac{a(a+k)(a+2k)\cdots(a+mk-2k)}{(a+mk-k)^{m-1}}$$
$$=C(a,k)-mC(a+mk-k,mk)+\frac{1}{k}\ln\frac{a+mk-k}{a}.$$

定理17′ 设 m 为正整数，且 $m\geqslant 2$，则
$$\sum_{n=1}^{\infty}\left(\frac{1}{mn-m+1}+\frac{1}{mn-m+2}+\frac{1}{mn-m+3}+\cdots\right.$$
$$\left.+\frac{1}{mn-1}-\frac{m-1}{mn}\right)=\ln m.$$

证明 根据命题1，$mC(m,m)=C(1,1)$。于定理17令 $a=1$，$k=1$，则有
$$\sum_{n=1}^{\infty}\left(\frac{1}{mn-m+1}+\frac{1}{mn-m+2}+\frac{1}{mn-m+3}\right.$$
$$\left.+\cdots+\frac{1}{mn-1}-\frac{m-1}{mn}\right)$$
$$=C(1,1)-mC(m,m)+\ln m=\ln m.$$

于定理17′，令$m=2,3,4,\cdots$则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\ln 2,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{2}{3n}\right)=\ln 3,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}+\frac{1}{4n-1}-\frac{3}{4n}\right)=\ln 4,$$

…

于定理17令 $m=3$，$a=3$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n}+\frac{1}{3n+1}-\frac{2}{3n+2}\right)=\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\ln 3;$$

令 $m=3$，$a=1$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}+\frac{1}{6n-3}-\frac{2}{6n-1}\right)=\frac{\pi\sqrt{3}}{4}-\frac{1}{4}\ln 3;$$

令　$m=3$，$a=1$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-7}-\frac{2}{12n-3}\right)=\frac{\pi}{4}+\frac{1}{4}\ln 3;$$

令　$m=3$，$a=5$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-19}+\frac{1}{24n-11}-\frac{2}{24n-3}\right)$$

$$=\frac{\pi}{8}+\frac{1}{8}\ln 3-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1);$$

令　$m=4$，$a=3$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}+\frac{1}{4n}+\frac{1}{4n+1}-\frac{3}{4n+2}\right)=\frac{1}{2};$$

令　$m=4$，$a=1$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}+\frac{1}{8n-5}+\frac{1}{8n-3}-\frac{3}{8n-1}\right)$$

$$=\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}-\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

令 $m=4$，$a=5$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-3}+\frac{1}{8n-1}+\frac{1}{8n+1}-\frac{3}{8n+3}\right)$$

$$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

令 $m=4$，$a=1$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-8}+\frac{1}{12n-5}-\frac{3}{12n-2}\right)=\frac{2\pi}{3\sqrt{3}};$$

令 $m=4$，$a=2$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-10}+\frac{1}{12n-7}+\frac{1}{12n-4}-\frac{3}{12n-1}\right)$$

$$=\frac{\pi}{3\sqrt{3}}+\frac{\pi}{3}-\frac{1}{3}\ln 2-\frac{1}{\sqrt{3}}\ln(2+\sqrt{3})。$$

仿照定理17的证法，即可证明以下一系列的定理。

定理18 设 m 为正整数，且 $m\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-3)+a}$$

$$\left.-\frac{m-1}{k(mn-2)+a}+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-2k,mk)$$

$$+\frac{1}{k}\ln\frac{a+mk-2k}{a}。$$

于定理18令 $a=k-1$，$m=k+1$，则有

定理18′ 设 k 为正整数，且 $k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k)-1}+\frac{1}{k(kn+n-k+1)-1}+\frac{1}{k(kn+n-k+2)-1}+\cdots+\frac{1}{k(kn+n-2)-1}-\frac{k}{k(kn+n-1)-1}+\frac{1}{k(kn+n)-1}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理18′令 $k=2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}-\frac{2}{6n-3}+\frac{1}{6n-1}\right)=\frac{1}{2}\ln 3,$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-3)(6n-1)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}-\frac{2}{6n-3}+\frac{1}{6n-1}\right)=\frac{1}{16}\ln 3;$$

令 $k=3$，4，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-10}+\frac{1}{12n-7}-\frac{3}{12n-4}+\frac{1}{12n-1}\right)$$

$$=\frac{2}{3}\ln 2;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{20n-17}+\frac{1}{20n-13}+\frac{1}{20n-9}-\frac{4}{20n-5}+\frac{1}{20n-1}\right)$$

$$=\frac{1}{4}\ln 5.$$

于定理18令 $m=3$，$a=1$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}-\frac{2}{3n-1}+\frac{1}{3n}\right)=\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\ln 3,$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)3n}$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}-\frac{2}{3n-1}+\frac{1}{3n}\right)$$

$$=\frac{\pi}{4\sqrt{3}}-\frac{1}{4}\ln 3;$$

令 $m=3$，$a=1$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{2}{12n-7}+\frac{1}{12n-3}\right)$$

$$=\frac{\pi\sqrt{3}}{8}-\frac{\pi}{8}-\frac{1}{8}\ln 3+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}),$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-7)(12n-3)}$

$$=\frac{1}{32}\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{2}{12n-7}+\frac{1}{12n-3}\right)$$

$$=\frac{\pi\sqrt{3}}{256}-\frac{\pi}{256}-\frac{1}{256}\ln 3+\frac{\sqrt{3}}{128}\ln(2+\sqrt{3});$$

令 $m=3$，$a=3$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-9}-\frac{2}{12n-5}+\frac{1}{12n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{8}+\frac{\pi}{8}-\frac{1}{8}\ln 3+\frac{\sqrt{3}}{4}\ln(2+\sqrt{3}),$$

于是 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(12n-9)(12n-5)(12n-1)}$

$$=\frac{1}{32}\sum_{n=1}^{\infty}\left(\frac{1}{12n-9}-\frac{2}{12n-5}+\frac{1}{12n-1}\right)$$

$$=-\frac{\pi\sqrt{3}}{256}+\frac{\pi}{256}-\frac{1}{256}\ln 3+\frac{\sqrt{3}}{128}\ln(2+\sqrt{3});$$

令 $m=3$，$a=1$，$k=8$，即可推得

欧拉常数？

$$\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{2}{24n-15}+\frac{1}{24n-7}\right)$$

$$=\frac{\pi}{8}+\frac{1}{8}\ln 3+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

于是 $$\sum_{n=1}^{\infty}\frac{1}{(24n-23)(24n-15)(24n-7)}$$

$$=\frac{1}{128}\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{2}{24n-15}+\frac{1}{24n-7}\right)$$

$$=\frac{\pi}{1024}+\frac{1}{1024}\ln 3+\frac{1}{256\sqrt{2}}\ln(\sqrt{2}+1);$$

令 $m=4$，$a=1$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{3}{4n-1}+\frac{1}{4n}\right)$$

$$=\frac{\pi}{2}-\ln 2;$$

令 $m=4$，$a=1$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}+\frac{1}{8n-5}-\frac{3}{8n-3}+\frac{1}{8n-1}\right)$$

$$=\frac{\pi}{2\sqrt{2}}-\frac{\pi}{4}+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

令 $m=4$，$a=1$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}+\frac{1}{12n-8}-\frac{3}{12n-5}+\frac{1}{12n-2}\right)$$

$$=-\frac{\pi}{3\sqrt{3}}+\frac{\pi}{3}-\frac{1}{3}\ln 2+\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}).$$

定理19 设m为正整数，且$m\geqslant 4$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-4)+a}$$

$$\left.-\frac{m-1}{k(mn-3)+a}+\frac{1}{k(mn-2)+a}+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-3k,mk)$$

$$+\frac{1}{k}\ln\frac{a+mk-3k}{a}.$$

于定理19令$a=k-1$，$m=2k+1$，则有

定理19′ 设k为正整数，且$k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(2kn+n-2k)-1}+\frac{1}{k(2kn+n-2k+1)-1}\right.$$

$$+\frac{1}{k(2kn+n-2k+2)-1}+\cdots+\frac{1}{k(2kn+n-3)-1}$$

$$-\frac{2k}{k(2kn+n-2)-1}+\frac{1}{k(2kn+n-1)-1}$$

$$\left.+\frac{1}{k(2kn+n)-1}\right]$$

$$=\frac{1}{k}\ln(2k+1).$$

于定理19′令$k=2,3$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}+\frac{1}{10n-7}-\frac{4}{10n-5}+\frac{1}{10n-3}+\frac{1}{10n-1}\right)$$

$$=\frac{1}{2}\ln 5;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{21n-19}+\frac{1}{21n-16}+\frac{1}{21n-13}+\frac{1}{21n-10}-\frac{6}{21n-7}\right.$$

$$\left.+\frac{1}{21n-4}+\frac{1}{21n-1}\right)$$

$$=\frac{1}{3}\ln 7.$$

于定理19令 $a=k-2$，$m=k+1$，则有

定理19″ 设 k 为正整数，且 $k\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k)-2}+\frac{1}{k(kn+n-k+1)-2}\right.$$

$$+\frac{1}{k(kn+n-k+2)-2}+\cdots+\frac{1}{k(kn+n-3)-2}$$

$$-\frac{k}{k(kn+n-2)-2}+\frac{1}{k(kn+n-1)-2}$$

$$\left.+\frac{1}{k(kn+n)-2}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理19″令 $k=3,4,5$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{3}{12n-8}+\frac{1}{12n-5}+\frac{1}{12n-2}\right)$$

$$=\frac{2}{3}\ln 2;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}+\frac{1}{10n-7}-\frac{2}{12n-5}+\frac{1}{10n-3}+\frac{1}{10n-1}\right)$$

$$=\frac{1}{2}\ln 5;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-27}+\frac{1}{30n-22}+\frac{1}{30n-17}-\frac{5}{30n-12}\right.$$

$$\left.+\frac{1}{30n-7}+\frac{1}{30n-2}\right)$$

$$=\frac{1}{5}\ln 6.$$

于定理19令 $m=4$，$a=2$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-10}-\frac{3}{12n-7}+\frac{1}{12n-4}+\frac{1}{12n-1}\right)$$

$$=\frac{\pi}{3\sqrt{3}}-\frac{\pi}{3}-\frac{1}{3}\ln 2+\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}).$$

定理20 设m为正整数，且$m\geqslant 5$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-5)+a}-\frac{m-1}{k(mn-4)+a}+\frac{1}{k(mn-3)+a}+\frac{1}{k(mn-2)+a}+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-4k,mk)+\frac{1}{k}\ln\frac{a+mk-4k}{a}.$$

于定理20令$a=k-1$，$m=3k+1$，则有

定理20′ 设k为正整数，且$k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(3kn+n-3k)-1}+\frac{1}{k(3kn+n-3k+1)-1}+\frac{1}{k(3kn+n-3k+2)-1}+\cdots+\frac{1}{k(3kn+n-4)-1}-\frac{3k}{k(3kn+n-3)-1}+\frac{1}{k(3kn+n-2)-1}+\frac{1}{k(3kn+n-1)-1}+\frac{1}{k(3kn+n)-1}\right]$$

$$=\frac{1}{k}\ln(3k+1).$$

于定理20′令$k=2,3$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{14n-13}+\frac{1}{14n-11}+\frac{1}{14n-9}-\frac{6}{14n-7}+\frac{1}{14n-5}+\frac{1}{14n-3}+\frac{1}{14n-1}\right)$$

$$=\frac{1}{2}\ln 7;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-28}+\frac{1}{30n-25}+\cdots+\frac{1}{30n-13}-\frac{1}{30n-10}+\frac{1}{30n-7}+\frac{1}{30n-4}+\frac{1}{30n-1}\right)$$

$$=\frac{1}{3}\ln 10.$$

于定理20令 $a=k-3$，$m=k+1$，则有

定理20″ 设 k 为正整数，且 $k\geqslant 4$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k)-3}+\frac{1}{k(kn+n-k+1)-3}+\frac{1}{k(kn+n-k+2)-3}+\cdots+\frac{1}{k(kn+n-4)-3}-\frac{k}{k(kn+n-3)-3}+\frac{1}{k(kn+n-2)-3}+\frac{1}{k(kn+n-1)-3}+\frac{1}{k(kn+n)-3}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理20″令 $k=4,5,6$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{20n-19}-\frac{4}{20n-15}+\frac{1}{20n-11}+\frac{1}{20n-7}+\frac{1}{20n-3}\right)$$

$$=\frac{1}{4}\ln 5;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-28}+\frac{1}{30n-23}-\frac{5}{30n-18}+\frac{1}{30n-13}+\frac{1}{30n-8}\right.$$

$$\left.+\frac{1}{30n-3}\right)$$

$$=\frac{1}{5}\ln 6;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{14n-13}+\frac{1}{14n-11}+\frac{1}{14n-9}-\frac{6}{14n-7}+\frac{1}{14n-5}\right.$$

$$\left.+\frac{1}{14n-3}+\frac{1}{14n-1}\right)$$

$$=\frac{1}{2}\ln 7.$$

于定理20令 $m=5$，$a=2$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(-\frac{1}{15n-13}+\frac{4}{15n-10}-\frac{1}{15n-7}-\frac{1}{15n-4}-\frac{1}{15n-1}\right)$$

$$=\frac{\pi}{3\sqrt{3}}-\frac{1}{3}\ln 5.$$

定理21 设m为正整数，且$m\geqslant 6$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-6)+a}$$

$$-\frac{m-1}{k(mn-5)+a}+\frac{1}{k(mn-4)+a}+\cdots$$

$$\left.+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-5k,mk)$$

$$+\frac{1}{k}\ln\frac{a+mk-5k}{a}.$$

于定理21令 $a=k-1$，$m=4k+1$，则有

定理21′ 设 k 为正整数，且 $k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(4kn+n-4k)-1}+\frac{1}{k(4kn+n-4k+1)-1}\right.$$

$$+\frac{1}{k(4kn+n-4k+2)-1}+\cdots+\frac{1}{k(4kn+n-5)-1}$$

$$-\frac{4k}{k(4kn+n-4)-1}+\frac{1}{k(4kn+n-3)-1}$$

$$\left.+\cdots+\frac{1}{k(4kn+n)-1}\right]$$

$$=\frac{1}{k}\ln(4k+1).$$

于定理21′令 $k=2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{18n-17}+\frac{1}{18n-15}+\frac{1}{18n-13}+\frac{1}{18n-11}-\frac{8}{18n-9}\right.$$

$$\left.+\frac{1}{18n-7}+\cdots+\frac{1}{18n-1}\right)$$

$$=\ln 3.$$

于定理21令 $a=k-2$，$m=2k+1$，则有

定理21″ 设 k 为正整数，且 $k\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(2kn+n-2k)-2}+\frac{1}{k(2kn+n-2k+1)-2}\right.$$

$$+\frac{1}{k(2kn+n-2k+2)-2}+\cdots+\frac{1}{k(2kn+n-5)-2}$$

$$-\frac{2k}{k(2kn+n-4)-2}+\frac{1}{k(2kn+n-3)-2}$$

$$\left.+\cdots+\frac{1}{k(2kn+n)-2}\right]$$

$$=\frac{1}{k}\ln(2k+1).$$

于定理21″令 $k=3,4$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{21n-20}+\frac{1}{21n-17}-\frac{6}{21n-14}+\frac{1}{21n-11}+\cdots+\frac{1}{21n-2}\right)$$

$$=\frac{1}{3}\ln 7;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{18n-17}+\frac{1}{18n-15}+\frac{1}{18n-13}+\frac{1}{18n-11}-\frac{8}{18n-9}+\frac{1}{18n-7}+\cdots+\frac{1}{18n-1}\right)$$

$$=\ln 3.$$

于定理21令 $a=k-4$，$k=k+1$，则有

定理21‴ 设 k 为正整数，且 $k\geqslant5$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k)-4}+\frac{1}{k(kn+n-k+1)-4}+\frac{1}{k(kn+n-k+2)-4}+\cdots+\frac{1}{k(kn+n-5)-4}-\frac{k}{k(kn+n-4)-4}+\frac{1}{k(kn+n-3)-4}+\cdots+\frac{1}{k(kn+n)-4}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理21‴令 $k=5,6$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-29}-\frac{5}{30n-24}+\frac{1}{30n-19}+\cdots+\frac{1}{30n-4}\right)$$

$$=\frac{1}{5}\ln 6;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{21n-20}+\frac{1}{21n-17}-\frac{6}{21n-14}+\frac{1}{21n-11}+\cdots\right.$$

$$\left.+\frac{1}{21n-2}\right)$$

$$=\frac{1}{3}\ln 7.$$

定理22 设m为正整数，且$m\geqslant 7$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-7)+a}$$

$$-\frac{m-1}{k(mn-6)+a}+\frac{1}{k(mn-5)+a}+\cdots$$

$$\left.+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-6k,mk)$$

$$+\frac{1}{k}\ln\frac{a+mk-6k}{a}.$$

于定理22令$a=k-1$，$m=5k+1$，则有

定理22′ 设k为正整数，且$k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(5kn+n-5k)-1}+\frac{1}{k(5kn+n-5k+1)-1}\right.$$

$$+\frac{1}{k(5kn+n-5k+2)-1}+\cdots+\frac{1}{k(5kn+n-6)-1}$$

$$-\frac{5k}{k(5kn+n-5)-1}+\frac{1}{k(5kn+n-4)-1}+\cdots$$

$$+\frac{1}{k(5kn+n)-1}\Bigg]$$

$$=\frac{1}{k}\ln(5k+1).$$

于定理22′令 $k=2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{22n-21}+\frac{1}{22n-19}+\cdots+\frac{1}{22n-13}-\frac{10}{22n-11}\right.$$

$$\left.+\frac{1}{22n-9}+\cdots+\frac{1}{22n-1}\right)$$

$$=\frac{1}{2}\ln 11.$$

于定理22令 $a=k-5$，$m=k+1$，则有

定理22″ 设 k 为正整数，且 $k\geqslant 6$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k)-5}+\frac{1}{k(kn+n-k+1)-5}\right.$$

$$+\frac{1}{k(kn+n-k+2)-5}+\cdots+\frac{1}{k(kn+n-6)-5}$$

$$-\frac{k}{k(kn+n-5)-5}+\frac{1}{k(kn+n-4)-5}$$

$$\left.+\cdots+\frac{1}{k(kn+n)-5}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理22″令 $k=6,7$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{42n-41}-\frac{6}{42n-35}+\frac{1}{42n-29}+\cdots+\frac{1}{42n-5}\right)$$

$$=\frac{1}{6}\ln 7;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{56n-54}+\frac{1}{56n-47}-\frac{7}{56n-40}+\frac{1}{56n-33}\right.$$

$$\left.+\cdots+\frac{1}{56n-5}\right)$$

$$=\frac{3}{7}\ln 2.$$

定理23 设m为正整数，且$m\geqslant 8$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\cdots+\frac{1}{k(mn-8)+a}$$

$$-\frac{m-1}{k(mn-7)+a}+\frac{1}{k(mn-6)+a}+\cdots$$

$$\left.+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+mk-7k,mk)$$

$$+\frac{1}{k}\ln\frac{a+mk-7k}{a}.$$

于定理23令$a=k-1$，$m=6k+1$。则有

定理23′ 设k为正整数，且$k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(6kn+n-6k)-1}+\frac{1}{k(6kn+n-6k+1)-1}\right.$$

$$+\frac{1}{k(6kn+n-6k+2)-1}+\cdots+\frac{1}{k(6kn+n-7)-1}$$

$$-\frac{6k}{k(6kn+n-6)-1}+\frac{1}{k(6kn+n-5)-1}+\cdots$$

$$\left.+\frac{1}{k(6kn+n)-1}\right]$$

$$=\frac{1}{k}\ln(6k+1).$$

于定理23′令 $k=2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{26n-25}+\frac{1}{26n-23}+\cdots+\frac{1}{26n-15}-\frac{12}{26n-13}+\frac{1}{26n-11}+\cdots+\frac{1}{26n-1}\right)$$

$$=\frac{1}{2}\ln 13.$$

于定理23令 $a=k-2$，$m=3k+1$，则有

定理23″ 设 k 为正整数，且 $k\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(3kn+n-3k)-2}+\frac{1}{k(3kn+n-3k+1)-2}+\frac{1}{k(3kn+n-3k+2)-2}+\cdots+\frac{1}{k(3kn+n-7)-2}-\frac{3k}{k(3kn+n-6)-2}+\frac{1}{k(3kn+n-5)-2}+\cdots+\frac{1}{k(3kn+n)-2}\right]$$

$$=\frac{1}{k}\ln(3k+1).$$

于定理23″令 $k=3,4$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-29}+\frac{1}{30n-26}+\frac{1}{30n-23}-\frac{9}{30n-20}+\frac{1}{30n-17}+\cdots+\frac{1}{30n-2}\right)$$

$$=\frac{1}{3}\ln 10,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{26n-25}+\frac{1}{26n-23}+\cdots+\frac{1}{26n-15}-\frac{12}{26n-13}\right.$$

$$+\frac{1}{26n-11}+\cdots+\frac{1}{26n-1}\Big)$$

$$=\frac{1}{2}\ln 13.$$

于定理23令 $a=k-3$，$m=2k+1$，则有

定理23‴ 设 k 为正整数，且 $k\geqslant 4$，则

$$\sum_{n=1}^{\infty}\Big[\frac{1}{k(2kn+n-2k)-3}+\frac{1}{k(2kn+n-2k+1)-3}$$

$$+\frac{1}{k(2kn+n-2k+2)-3}+\cdots+\frac{1}{k(2kn+n-7)-3}$$

$$-\frac{2k}{k(2kn+n-6)-3}+\frac{1}{k(2kn+n-5)-3}$$

$$+\cdots+\frac{1}{k(2kn+n)-3}\Big]$$

$$=\frac{1}{k}\ln(2k+1).$$

于定理23‴令 $k=4$，则有

$$\sum_{n=1}^{\infty}\Big(\frac{1}{36n-35}+\frac{1}{36n-31}-\frac{8}{36n-27}+\frac{1}{36n-23}$$

$$+\cdots+\frac{1}{36n-3}\Big)$$

$$=\frac{1}{2}\ln 3.$$

于定理23令 $a=k-6$，$m=k+1$，则有

定理23⁽⁴⁾ 设 k 为正整数，且 $k\geqslant 7$，则

$$\sum_{n=1}^{\infty}\Big[\frac{1}{k(kn+n-k)-6}+\frac{1}{k(kn+n-k+1)-6}$$

$$+\frac{1}{k(kn+n-k+2)-6}+\cdots+\frac{1}{k(kn+n-7)-6}$$

$$-\frac{k}{k(kn+n-6)-6}+\frac{1}{k(kn+n-5)-6}+\cdots$$

$$+\frac{1}{k(kn+n)-6}\Bigg]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理23(4)令 $k=7$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{56n-55}-\frac{7}{56n-48}+\frac{1}{56n-41}+\cdots+\frac{1}{56n-6}\right)$$

$$=\frac{3}{7}\ln 2.$$

类似定理17——23这样的结果，可以给出任意多个.

定理24 设 m 为正整数，且 $m\geqslant 2$，则

$$\sum_{n=1}^{\infty}\Bigg[\frac{1}{k(mn-m)+a}-\frac{m-1}{k(mn-m+1)+a}$$

$$+\frac{1}{k(mn-m+2)+a}+\frac{1}{k(mn-m+3)+a}$$

$$+\cdots+\frac{1}{k(mn-1)+a}\Bigg]$$

$$=C(a,k)-mC(a+k,mk)+\frac{1}{k}\ln\frac{a+k}{a}.$$

于定理24令 $a=1$，$m=k+1$，则有

定理24′ 设 k 为正整数，则

$$\sum_{n=1}^{\infty}\Bigg[\frac{1}{k(kn+n-k-1)+1}-\frac{1}{k(kn+n-k)+1}$$

$$+\frac{1}{k(kn+n-k+1)+1}+\frac{1}{k(kn+n-k+2)+1}$$

$$+\cdots+\frac{1}{k(kn+n-1)+1}\Bigg]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理24′令 $k=1,2,3$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-3)(6n-1)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}-\frac{2}{6n-3}+\frac{1}{6n-1}\right)$$

$$=\frac{1}{16}\ln 3,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{3}{12n-8}+\frac{1}{12n-5}+\frac{1}{12n-2}\right)$$

$$=\frac{2}{3}\ln 2.$$

于定理24令 $m=4$，$a=1$，$k=1$。即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{3}{4n-2}+\frac{1}{4n-1}+\frac{1}{4n}\right)=0;$$

令 $m=4$，$a=1$，$k=2$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{3}{8n-5}+\frac{1}{8n-3}+\frac{1}{8n-1}\right)$$

$$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{4}+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1);$$

令 $m=4$，$a=2$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-10}-\frac{3}{12n-7}+\frac{1}{12n-4}+\frac{1}{12n-1}\right)$$

$$=\frac{\pi}{3\sqrt{3}}-\frac{\pi}{3}-\frac{1}{3}\ln 2+\frac{1}{\sqrt{3}}\ln(2+\sqrt{3});$$

令$m=8$，$a=1$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\left(-\frac{1}{8n-7}+\frac{7}{8n-6}-\frac{1}{8n-5}-\cdots-\frac{1}{8n}\right)=\frac{\pi}{2}.$$

定理25 设m为正整数，且$m\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$
$$-\frac{m-1}{k(mn-m+2)+a}+\frac{1}{k(mn-m+3)+a}$$
$$\left.+\frac{1}{k(mn-m+4)+a}+\cdots+\frac{1}{k(mn-1)+a}\right]$$
$$=C(a,k)-mC(a+2k,mk)+\frac{1}{k}\ln\frac{a+2k}{a}.$$

于定理25令$a=1$，$m=2k+1$，则有

定理25′ 设k为正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(2kn+n-2k-1)+1}+\frac{1}{k(2kn+n-2k)+1}\right.$$
$$-\frac{2k}{k(2kn+n-2k+1)+1}+\frac{1}{k(2kn+n-2k+2)+1}$$
$$\left.+\cdots+\frac{1}{k(2kn+n-1)+1}\right]$$
$$=\frac{1}{k}\ln(2k+1).$$

于定理25′令$k=1,2,3$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{2}{3n}\right)$$
$$=\ln 3,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}+\frac{1}{10n-7}-\frac{4}{10n-5}+\frac{1}{10n-3}+\frac{1}{10n-1}\right)$$

$$=\frac{1}{2}\ln 5,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{21n-20}+\frac{1}{21n-17}-\frac{6}{21n-14}+\frac{1}{21n-11}\right.$$

$$\left.+\cdots+\frac{1}{21n-2}\right)$$

$$=\frac{1}{3}\ln 7.$$

于定理25令 $a=2$，$m=k+1$，则有

定理25″ 设 k 为正整数，且 $k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k-1)+2}+\frac{1}{k(kn+n-k)+2}\right.$$

$$-\frac{k}{k(kn+n-k+1)+2}+\frac{1}{k(kn+n-k+2)+2}$$

$$\left.+\cdots+\frac{1}{k(kn+n-1)+2}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理25″令 $k=2,3,4$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{2}{3n}\right)$$

$$=\ln 3,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{12n-10}+\frac{1}{12n-7}-\frac{3}{12n-4}+\frac{1}{12n-1}\right)$$

$$=\frac{2}{3}\ln 2,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}+\frac{1}{10n-7}-\frac{4}{10n-5}+\frac{1}{10n-3}+\frac{1}{10n-1}\right)$$

$$=\frac{1}{2}\ln 5.$$

定理26 设 m 为正整数，且 $m\geqslant 4$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}+\frac{1}{k(mn-m+2)+a}-\frac{m-1}{k(mn-m+3)+a}+\frac{1}{k(mn-m+4)+a}+\cdots+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,k)-mC(a+3k,mk)+\frac{1}{k}\ln\frac{a+3k}{a}.$$

于定理26令 $a=1$，$m=3k+1$，则有

定理26′ 设 k 为正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(3kn+n-3k-1)+1}+\frac{1}{k(3kn+n-3k)+1}+\frac{1}{k(3kn+n-3k+1)+1}-\frac{3k}{k(3kn+n-3k+2)+1}+\frac{1}{k(3kn+n-3k+3)+1}+\cdots+\frac{1}{k(3kn+n-1)+1}\right]$$

$$=\frac{1}{k}\ln(3k+1).$$

于定理26′令 $k=1,2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}+\frac{1}{4n-1}-\frac{3}{4n}\right)$$

$$=\ln 4,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{14n-13}+\frac{1}{14n-11}+\frac{1}{14n-9}-\frac{6}{14n-7}+\frac{1}{14n-5}+\frac{1}{14n-3}+\frac{1}{14n-1}\right)$$

$$=\frac{1}{2}\ln 7.$$

于定理26令 $a=3$，$m=k+1$，则有

定理26″ 设 k 为正整数，且 $k\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k-1)+3}+\frac{1}{k(kn+n-k)+3}\right.$$
$$+\frac{1}{k(kn+n-k+1)+3}-\frac{k}{k(kn+n-k+2)+3}$$
$$\left.+\frac{1}{k(kn+n-k+3)+3}+\cdots+\frac{1}{k(kn+n-1)+3}\right]$$
$$=\frac{1}{k}\ln(k+1).$$

于定理26″令 $k=3,4,5$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}+\frac{1}{4n-1}-\frac{3}{4n}\right)$$
$$=\ln 4,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{20n-17}+\frac{1}{20n-13}+\frac{1}{20n-9}-\frac{4}{20n-5}+\frac{1}{20n-1}\right)$$
$$=\frac{1}{4}\ln 5,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-27}+\frac{1}{30n-22}+\frac{1}{30n-17}-\frac{5}{30n-12}\right.$$
$$\left.+\frac{1}{30n-7}+\frac{1}{30n-2}\right)$$
$$=\frac{1}{5}\ln 6.$$

于定理26令 $m=5$，$a=1$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{15n-14}+\frac{1}{15n-11}+\frac{1}{15n-8}-\frac{4}{15n-5}+\frac{1}{15n-2}\right)$$

$$=\frac{\pi}{3\sqrt{3}}+\frac{1}{3}\ln 5;$$

令$m=5$，$a=1$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{40n-39}+\frac{1}{40n-31}+\frac{1}{40n-23}-\frac{4}{40n-15}+\frac{1}{40n-7}\right)$$

$$=\frac{\pi}{4\sqrt{2}}+\frac{1}{8}\ln 5+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1),$$

令$m=7$，$a=11$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{56n-45}+\frac{1}{56n-37}+\frac{1}{56n-29}-\frac{6}{56n-21}+\frac{1}{56n-13}\right.$$

$$\left.+\frac{1}{56n-5}+\frac{1}{56n+3}\right)$$

$$=\frac{\pi}{4\sqrt{2}}-\frac{\pi}{8}+\frac{1}{8}\ln 7-\frac{1}{3}.$$

定理27　设m为正整数，且$m\geqslant 5$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}\right.$$

$$+\frac{1}{k(mn-m+2)+a}+\frac{1}{k(mn-m+3)+a}$$

$$-\frac{m-1}{k(mn-m+4)+a}+\frac{1}{k(mn-m+5)+a}$$

$$\left.+\cdots+\frac{1}{k(mn-1)+a}\right]$$

$$=C(a,m)-mC(a+4k,mk)+\frac{1}{k}\ln\frac{a+4k}{a}.$$

于定理27令$a=1$，$m=4k+1$，则有

定理27′　设k为正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(4kn+n-4k-1)+1}+\frac{1}{k(4kn+n-4k)+1}\right.$$

$$+\frac{1}{k(4kn+n-4k+1)+1}+\frac{1}{k(4kn+n-4k+2)+1}$$

$$-\frac{4k}{k(4kn+n-4k+3)+1}+\frac{1}{k(4kn+n-4k+4)+1}$$

$$\left.+\cdots+\frac{1}{k(4kn+n-1)+1}\right]$$

$$=\frac{1}{k}\ln(4k+1).$$

于定理27′ 令 $k=1,2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{5n-4}+\frac{1}{5n-3}+\frac{1}{5n-2}+\frac{1}{5n-1}-\frac{4}{5n}\right)$$

$$=\ln 5;$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{18n-17}+\frac{1}{18n-15}+\frac{1}{18n-13}+\frac{1}{18n-11}-\frac{8}{18n-9}\right.$$

$$\left.+\frac{1}{18n-7}+\cdots+\frac{1}{18n-1}\right)$$

$$=\ln 3.$$

于定理27令 $a=2$，$m=2k+1$，则有

定理27″ 设 k 为正整数，且 $k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(2kn+n-2k-1)+2}+\frac{1}{k(2kn+n-2k)+2}\right.$$

$$+\frac{1}{k(2kn+n-2k+1)+2}+\frac{1}{k(2kn+n-2k+2)+2}$$

$$-\frac{2k}{k(2kn+n-2k+3)+2}+\frac{1}{k(2kn+n-2k+4)+2}$$

$$\left.+\cdots+\frac{1}{k(2kn+n-1)+2}\right]$$

$$=\frac{1}{k}\ln(2k+1).$$

于定理27″令 $k=2,3$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{5n-4}+\frac{1}{5n-3}+\frac{1}{5n-2}+\frac{1}{5n-1}-\frac{4}{5n}\right)$$
$$=\ln 5,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{21n-19}+\frac{1}{21n-16}+\frac{1}{21n-13}+\frac{1}{21n-10}-\frac{6}{21n-7}\right.$$
$$\left.+\frac{1}{21n-4}+\frac{1}{21n-1}\right)$$
$$=\frac{1}{3}\ln 7.$$

于定理27令 $a=5$，$m=k+1$，则有

定理27‴ 设 k 为正整数，且 $k\geqslant4$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k-1)+4}+\frac{1}{k(kn+n-k)+4}\right.$$
$$+\frac{1}{k(kn+n-k+1)+4}+\frac{1}{k(kn+n-k+2)+4}$$
$$-\frac{k}{k(kn+n-k+3)+4}+\frac{1}{k(kn+n-k+4)+4}$$
$$\left.+\cdots+\frac{1}{k(kn+n-1)+4}\right]$$
$$=\frac{1}{k}\ln(k+1).$$

于定理27‴，令 $k=4,5$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{5n-4}+\frac{1}{5n-3}+\frac{1}{5n-2}+\frac{1}{5n-1}-\frac{4}{5n}\right)$$
$$=\ln 5,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{36n-26}+\frac{1}{36n-21}+\frac{1}{36n-16}+\frac{1}{36n-11}-\frac{5}{30n-6}\right.$$

$$+\frac{1}{30n-1}\Big)=\frac{1}{5}\ln 6.$$

于定理27令 $m=5$，$a=3$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{40n-37}+\frac{1}{40n-29}+\frac{1}{40n-21}+\frac{1}{40n-13}-\frac{4}{40n-5}\right)$$

$$=\frac{\pi}{4\sqrt{2}}+\frac{1}{8}\ln 5-\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1);$$

令 $m=7$，$a=3$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\Big(\frac{1}{56n-53}+\frac{1}{56n-45}+\frac{1}{56n-37}+\frac{1}{56n-29}-\frac{6}{56n-21}$$

$$+\frac{1}{56n-13}+\frac{1}{56n-5}\Big)$$

$$=\frac{\pi}{4\sqrt{2}}-\frac{\pi}{8}+\frac{1}{8}\ln 7.$$

定理28 设 m 为正整数，且 $m\geqslant 6$，则

$$\sum_{n=1}^{\infty}\Big[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}+\cdots$$

$$+\frac{1}{k(mn-m+4)+a}-\frac{m-1}{k(mn-m+5)+a}$$

$$+\frac{1}{k(mn-m+6)+a}+\cdots+\frac{1}{k(mn-1)+a}\Big]$$

$$=C(a,k)-mC(a+5k,mk)+\frac{1}{k}\ln\frac{a+5k}{a}.$$

于定理28令 $a=1$，$m=5k+1$，则有

定理28′ 设 k 为正整数，则

$$\sum_{n=1}^{\infty}\Big[\frac{1}{k(5kn+n-5k-1)+1}+\frac{1}{k(5kn+n-5k)+1}+\cdots$$

$$+\frac{1}{k(5kn+n-5k+3)+1}-\frac{5k}{k(5kn+n-5k+4)+1}$$

$$+\frac{1}{k(5kn+n-5k+5)+1}+\cdots+\frac{1}{k(5kn+n-1)+1}\Bigg]$$

$$=\frac{1}{k}\ln(6k+1).$$

于定理28′令 $k=1$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}+\frac{1}{6n-4}+\frac{1}{6n-3}+\frac{1}{6n-2}+\frac{1}{6n-1}-\frac{5}{6n}\right)$$

$$=\ln 6.$$

于定理28令 $a=5$，$m=k+1$，则有

定理28″ 设 k 为正整数，且 $k\geqslant 5$，则

$$\sum_{n=1}^{\infty}\Bigg[\frac{1}{k(kn+n-k-1)+5}+\frac{1}{k(kn+n-k)+5}+\cdots$$

$$+\frac{1}{k(kn+n-k+3)+5}-\frac{k}{k(kn+n-k+4)+5}$$

$$+\frac{1}{k(kn+n-k+5)+5}+\cdots+\frac{1}{k(kn+n-1)+5}\Bigg]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理28″令 $k=5,6$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}+\frac{1}{6n-4}+\frac{1}{6n-3}+\frac{1}{6n-2}+\frac{1}{6n-1}-\frac{5}{6n}\right)$$

$$=\ln 6,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{42n-37}+\frac{1}{42n-31}+\cdots+\frac{1}{42n-13}-\frac{6}{42n-7}\right.$$

$$\left.+\frac{1}{42n-1}\right)$$

$$=\frac{1}{6}\ln 7.$$

于定理28令$m=7$，$a=1$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{28n-27}+\frac{1}{28n-23}+\cdots+\frac{1}{28n-11}-\frac{6}{28n-7}\right.$$
$$\left.+\frac{1}{28n-3}\right)$$
$$=\frac{\pi}{4}+\frac{1}{4}\ln 7.$$

定理29　设m为正整数，且$m\geqslant 7$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(mn-m)+a}+\frac{1}{k(mn-m+1)+a}+\cdots\right.$$
$$+\frac{1}{k(mn-m+5)+a}-\frac{m-1}{k(mn-m+6)+a}$$
$$\left.+\frac{1}{k(mn-m+7)+a}+\cdots+\frac{1}{k(mn-1)+a}\right]$$
$$=C(a,k)-mC(a+6k,mk)+\frac{1}{k}\ln\frac{a+6k}{a}.$$

于定理29令$a=1$，$m=6k+1$，则有

定理29′　设k为正整数，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(6kn+n-6k-1)+1}+\frac{1}{k(6kn+n-6k)+1}+\cdots\right.$$
$$+\frac{1}{k(6kn+n-6k+4)+1}-\frac{6k}{k(6kn+n-6k+5)+1}$$
$$\left.+\frac{1}{k(6kn+n-6k+6)+1}+\cdots+\frac{1}{k(6kn+n-1)+1}\right]$$
$$=\frac{1}{k}\ln(6k+1).$$

于定理29′令$k=1,2$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{7n-6}+\frac{1}{7n-5}+\cdots+\frac{1}{7n-1}-\frac{6}{7n}\right)$$

$=\ln 7$,

$$\sum_{n=1}^{\infty}\left(\frac{1}{26n-25}+\frac{1}{26n-23}+\cdots+\frac{1}{26n-15}-\frac{12}{26n-13}\right.$$
$$\left.+\frac{1}{26n-11}+\cdots+\frac{1}{26n-1}\right)$$
$$=\frac{1}{2}\ln 13.$$

于定理29令 $a=2$，$m=3k+1$，则有

定理29″ 设 k 为正整数，且 $k\geqslant 2$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(3kn+n-3k-1)+2}+\frac{1}{k(3kn+n-3k)+2}+\cdots\right.$$
$$+\frac{1}{k(3kn+n-3k+4)+2}-\frac{3k}{k(3kn+n-3k+5)+2}$$
$$\left.+\frac{1}{k(3kn+n-3k+6)+2}+\cdots+\frac{1}{k(3kn+n-1)+2}\right]$$
$$=\frac{1}{k}\ln(3k+1).$$

于定理29″令 $k=2,3$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{7n-6}+\frac{1}{7n-5}+\cdots+\frac{1}{7n-1}-\frac{6}{7n}\right)$$
$$=\ln 7,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{30n-28}+\frac{1}{30n-25}+\cdots+\frac{1}{30n-13}-\frac{9}{30n-10}\right.$$
$$\left.+\frac{1}{30n-7}+\frac{1}{30n-4}+\frac{1}{30n-1}\right)$$
$$=\frac{1}{3}\ln 10.$$

于定理29令 $a=3$，$m=2k+1$，则有

定理29‴ 设 k 为正整数，且 $k\geqslant 3$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(2kn+n-2k-1)+3}+\frac{1}{k(2kn+n-2k)+3}+\cdots\right.$$

$$+\frac{1}{k(2kn+n-2k+4)+3}-\frac{2k}{k(2kn+n-2k+5)+3}$$

$$\left.+\frac{1}{k(2kn+n-2k+6)+3}+\cdots+\frac{1}{k(2kn+n-1)+3}\right]$$

$$=\frac{1}{k}\ln(2k+1).$$

于定理29‴令 $k=4$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{36n-33}+\frac{1}{36n-29}+\cdots+\frac{1}{36n-13}-\frac{8}{36n-9}\right.$$

$$\left.+\frac{1}{36n-5}+\frac{1}{36n-1}\right)$$

$$=\frac{1}{2}\ln 3.$$

于定理29令 $a=6$，$m=k+1$，则有

定理29⁽⁴⁾ 设 k 为正整数，且 $k\geqslant 6$，则

$$\sum_{n=1}^{\infty}\left[\frac{1}{k(kn+n-k-1)+6}+\frac{1}{k(kn+n-k)+6}+\cdots\right.$$

$$+\frac{1}{k(kn+n-k+4)+6}-\frac{k}{k(kn+n-k+5)+6}$$

$$\left.+\frac{1}{k(kn+n-k+6)+6}+\cdots+\frac{1}{k(kn+n-1)+6}\right]$$

$$=\frac{1}{k}\ln(k+1).$$

于定理29⁽⁴⁾令 $k=7$，则有

$$\sum_{n=1}^{\infty}\left(\frac{1}{56n-50}+\frac{1}{56n-43}+\cdots+\frac{1}{56n-15}-\frac{7}{56n-8}\right.$$

$$+\frac{1}{56n-1}\Big)$$

$$=\frac{3}{7}\ln 2.$$

于定理29令 $m=7$，$a=1$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{56n-55}+\frac{1}{56n-47}+\cdots+\frac{1}{56n-15}-\frac{6}{56n-7}\right)$$

$$=\frac{\pi}{4\sqrt{2}}+\frac{\pi}{8}+\frac{1}{8}\ln 7.$$

类似定理24——29这样的结果，可以给出任意多个.

定理30 设 m 为正整数，则

$$\sum_{n=1}^{\infty}\Big[\frac{1}{k(2mn+n-2m-1)+a}+\frac{1}{k(2mn+n-2m)+a}$$

$$+\frac{1}{k(2mn+n-2m+1)+a}+\cdots$$

$$+\frac{1}{k(2mn+n-m-2)+a}-\frac{2m}{k(2mn+n-m-1)+a}$$

$$+\frac{1}{k(2mn+n-m)+a}+\frac{1}{k(2mn+n-m+1)+a}$$

$$+\cdots+\frac{1}{k(2mn+n-1)+a}\Big]$$

$$=C(a,k)-(2m+1)C(a+mk,2mk+k)$$

$$+\frac{1}{k}\ln\frac{a+mk}{a}.$$

于定理30令 $a=1$，$k=2$，则有

定理30′ 设 m 为正整数，则

$$\sum_{n=1}^{\infty}\Big[\frac{1}{(4m+2)n-4m-1}+\frac{1}{(4m+2)n-4m+1}$$

$$+\frac{1}{(4m+2)n-4m+3}+\cdots+\frac{1}{(4m+2)n-2m-3}$$

$$-\frac{2m}{(4m+2)n-2m-1}+\frac{1}{(4m+2)n-2m+1}$$

$$+\frac{1}{(4m+2)n-2m+3}+\cdots+\frac{1}{(4m+2)n-1}\Bigg]$$

$$=\frac{1}{2}\ln(2m+1).$$

于定理30′令 $m=1,2$，则有

$$\sum_{n=1}^{\infty}\frac{1}{(6n-5)(6n-3)(6n-1)}$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{6n-5}-\frac{2}{6n-3}+\frac{1}{6n-1}\right)$$

$$=\frac{1}{16}\ln 3,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{10n-9}+\frac{1}{10n-7}-\frac{4}{10n-5}+\frac{1}{10n-3}+\frac{1}{10n-1}\right)$$

$$=\frac{1}{2}\ln 5.$$

于定理30令 $m=1$，$a=1$，$k=1$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)3n}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}-\frac{2}{3n-1}+\frac{1}{3n}\right)$$

$$=\frac{\pi}{4\sqrt{3}}-\frac{1}{4}\ln 3;$$

令 $m=1$，$a=1$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(12n-11)(12n-7)(12n-3)}$$

$$=\frac{1}{32}\sum_{n=1}^{\infty}\left(\frac{1}{12n-11}-\frac{2}{12n-7}+\frac{1}{12n-3}\right)$$

$$=\frac{\pi\sqrt{3}}{256}-\frac{\pi}{256}-\frac{1}{256}\ln 3+\frac{\sqrt{3}}{128}\ln(2+\sqrt{3});$$

令$m=1$，$a=1$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(24n-23)(24n-15)(24n-7)}$$

$$=\frac{1}{128}\sum_{n=1}^{\infty}\left(\frac{1}{24n-23}-\frac{2}{24n-15}+\frac{1}{24n-7}\right)$$

$$=\frac{\pi}{1024}+\frac{1}{1024}\ln 3+\frac{1}{256\sqrt{2}}\ln(\sqrt{2}+1);$$

令$m=1$，$a=7$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\frac{1}{(24n-17)(24n-9)(24n-1)}$$

$$=\frac{1}{128}\sum_{n=1}^{\infty}\left(\frac{1}{24n-17}-\frac{2}{24n-9}+\frac{1}{24n-1}\right)$$

$$=-\frac{\pi}{1024}+\frac{1}{1024}\ln 3+\frac{1}{256\sqrt{2}}\ln(\sqrt{2}+1);$$

令$m=2$，$a=4$，$k=3$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{15n-11}+\frac{1}{15n-8}-\frac{4}{15n-5}+\frac{1}{15n-2}+\frac{1}{15n+1}\right)$$

$$=\frac{\pi}{3\sqrt{3}}+\frac{1}{3}\ln 5-1;$$

令$m=2$，$a=7$，$k=4$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{20n-13}+\frac{1}{20n-9}-\frac{4}{20n-5}+\frac{1}{20n-1}+\frac{1}{20n+3}\right)$$

$$=\frac{1}{4}\ln 5-\frac{1}{3};$$

令$m=2$，$a=9$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{40n-31}+\frac{1}{40n-23}-\frac{4}{40n-15}+\frac{1}{40n-7}+\frac{1}{40n+1}\right)$$

$$=\frac{\pi}{4\sqrt{2}}+\frac{1}{8}\ln 5+\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)-1;$$

令$m=2$，$a=1$，$k=12$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{60n-59}+\frac{1}{60n-47}-\frac{4}{60n-35}+\frac{1}{60n-23}+\frac{1}{60n-11}\right)$$

$$=\frac{\pi}{4\sqrt{3}}+\frac{1}{12}\ln 5+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3});$$

令$m=2$，$a=11$，$k=12$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{60n-49}+\frac{1}{60n-37}-\frac{4}{60n-25}+\frac{1}{60n-13}+\frac{1}{60n-1}\right)$$

$$=-\frac{\pi}{4\sqrt{3}}+\frac{1}{12}\ln 5+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3});$$

令$m=3$，$a=11$，$k=8$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{56n-45}+\frac{1}{56n-37}+\frac{1}{56n-29}-\frac{6}{56n-21}+\frac{1}{56n-13}+\frac{1}{56n-5}+\frac{1}{56n+3}\right)$$

$$=\frac{\pi}{4\sqrt{2}}-\frac{\pi}{8}+\frac{1}{8}\ln 7-\frac{1}{3};$$

令$m=3$，$a=13$，$k=12$，即可推得

$$\sum_{n=1}^{\infty}\left(\frac{1}{84n-71}+\frac{1}{84n-59}+\frac{1}{84n-47}-\frac{6}{84n-35}+\frac{1}{84n-23}+\frac{1}{84n-11}+\frac{1}{84n+1}\right)$$

$$=\frac{\pi}{6}+\frac{1}{12}\ln 7+\frac{1}{2\sqrt{3}}\ln(2+\sqrt{3})-1.$$

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